การวิเคราะห์และประยุกต์ระบบพิกัดเทอร์เรียน เพื่อใช้กับการเคลื่อนที่ของบรรยากาศ

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บทคัดย่อ

การแปลงจากระบบพิกัดฉากไปสู่ระบบพิกัดเทอร์เรียนโดยการใช้เทนเซอร์ สามารถใช้ในการสร้างแบบจำลองไฮโดรสแตติกที่มีสเกลขนาดเล็กและขนาดใหญ่ได้ ซึ่งสรุปได้ดังนี้

1. ใช้ทฤษฎีลูกโซ่กับสมการไฮโดรสแตติก ก่อนการแปลงจากระบบพิกัดฉากไปสู่ระบบพิกัดเทอร์เรียน ซึ่งข้อสำคัญก็คือ ในระบบพิกัดเทอร์เรียนนี้จะต้องมีความลาดเอียงน้อยกว่า 45°

2. การเปลี่ยนแปลงของเมตริกเทนเซอร์ของคริดวอลลุ่มจะหาผลลัพธ์ได้จากการหาค่าเฉลี่ยจากคริดวอลลุ่ม ซึ่งสามารถนำไปใช้ได้ทั้งแบบจำลองทางอุตุนิยมวิทยาประเภทนอนไฮโดรสแตติคและไฮโดรสแตติค

The Derivation Analysis and Application of Terrian-Following Coordinate Representation of Atmospheric Motion

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Abstract

This article uses tensor transformation procedures in order to derive a terrian-following coordinate system that is frequently used in a number of regional and mesoscale hydrostatic models. Tensor transformation procedures are used so a- to ensure physical invariance of the primitive equations between the Cartesian and terrian following systems. Among the major conclusions are as follows:

1. Applying the chain rule the hydrostatic equation, before transformation from a Cartesian to a terrian-following coordinate system, yields a different set of equations than if the hydrostatic assumption is applied after the tensor transformation is made. The hydrostatic equations in the two terrian-following representations are the same only when the slope of the terrian in the model is much less than 45°.

2. Variations of the metric tensor across a grid volume appear in the set of conservation equations as a result of averaging the equations over a grid voume. Such derivations have always been ignored in existing non-hydrostatic and hydrostatic meteorological models.

Introduction

The use of a terrain-following coordinate system in meteorological modeling was first introduced by Phillips(1975), and it has since been shown to be an effective mathematical representation. This concept of defining a coordinate surface coincident with the bottom topography permits more efficient use of computer resources, and it simplifies the application of lower boundary conditions. In Philips' original form, adopted by many models [e.g., the US Weather Service forecast models, Rieck(1979)], pressure is used to define the independent vertical coordinate σ, where surface pressure is used as the lower boundary.
For example, defines $\sigma = p / p_s$, where p_s is the surface pressure while p is the pressure at any level. For this example σ = 1 corresponds to the ground surface.

In recent years, σ has often been defined as a function of the height rather than pressure. This is advantageous because p_s is a function of time, whereas terrain height is not. The general form of the coordinate system transformation is given as

$$\sigma = s\frac{z - z_G}{s - z_G} \tag{1}$$

Where s is usually defined as a constant (generally defined as the top of the model) while z_G is the terrain height. The variable z is height, while σ is referred to as a generalized vertical coordinate. This form of a terrain-following coordinate has been used in recent years in regional and mesoscale models in which the hydrostatic assumption has been applied. In developing their equations, however, these investigators have applied the chain rule separately in the vertical and horizontal dimensions (utilizing the hydrostatic relation). Using (1), this results in the transformed hydrostatic equation given as

$$\frac{\partial\pi}{\partial\sigma} = -\frac{s - z_G}{s}\frac{g}{\theta} \tag{2}$$

where $\pi = c_p T / \theta$. This is appropriate if the hydrostatic assumption is exactly satisfied. However, the invariance of the physical representation (which must be retained, regardless of the coordinate formulation) is lost if the assumption is not exact, as discussed by Dutton (1976, p. 252). On the synoptic scale, in which horizontal scales are always much larger than the vertical scales of motion,this requirement is very closely satisfied.

On the mesoscale, however, it may be more appropriate to perform a tensor transformation of all three components of the equation of motion, before making the hydrostatic assumption. For use in a non-hydrostatic model , a rigorous transformation between coordinate system requires use of the properties of tensor analytsis in order to assure that the invariance of the physical representation is retained in all coordinate systems.

We perform a tensor transformation of the equations of motion and then apply the hydrostatic assumption. The resultant equations reduce to the form when certain simplifying assumptions are made. Moreover, since the hydrostatic assumption is applied later in the derivation of the transformed equations, a more in-depth understanding of the coordinate transformation is obtained.

The equation of motion

Dutton (1976) demonstrated that the contravariant form of the equation of motion in a generalized coordinate system, derived from the rectangular x-y-z (x^i) system, can be written as

$$\frac{\partial \tilde{u}^i}{\partial t} + \tilde{u}^j \tilde{u}^i_{;j} = -\tilde{G}^{ij}\theta \frac{\partial \pi}{\partial \tilde{x}^j} - g\frac{\partial \tilde{x}^i}{\partial z} - 2\tilde{\varepsilon}^{ijl}\tilde{\Omega}_j \tilde{u}_l \tag{3}$$

where $\tilde{u}^i$ is the contravariant component of velocity, $\tilde{G}^{ij}$ the contravariant metric tensor, $\tilde{x}^j$ represents the independent variable in the new coordinate system, and

$$\tilde{\varepsilon}^{ijl} = \varepsilon_{ijl}\tilde{G}^{-1/2}$$

$$\tilde{u}^i_{;j} = \frac{\partial \tilde{u}^i}{\partial \tilde{x}^j} + \tilde{\Gamma}^i_{jl}\tilde{u}^l$$

The term $g(\partial \tilde{x}^i/\partial z)$ is obtained from $\tilde{G}^{ij}\partial \Phi/\partial \tilde{x}^j$. The tilde is used to indicate a variable in the transformed coordinate system, while $\varepsilon_{ijl} = \varepsilon^{ijl}$ in the Cartesian system. The tensor ε_{ijl} is defined as zero if any two of the indices are equal, +1. If an even permutation of the indices occur, and -1 with an odd permutation. The parameter $\tilde{G}$ is the determinant of the contravariant form of the metric tensor $\tilde{G}^{ij}$ while $\tilde{\Gamma}^{i}_{jl}$ called the Christoffel symnbol, is given by

$$\tilde{\Gamma}^{i}_{jl} = \frac{\partial \tilde{x}^{i}}{\partial x^{m}} \frac{\partial^{2} x^{m}}{\partial \tilde{x}^{j} \partial \tilde{x}^{l}}$$

Eq (3) is somewhat cumbersome to work. However, it is essential to retain all the terms arise from the transformation in order to present tensor variance

When applying these equations to simulate meteorological systems, only the vertical coordinate to the rectangular system customarily transformed. In addition, it is necessary to average the transformed equations since (3) is only valid over spatial and temporal intervals which are much smaller than the mesoscale space and time scales used in meterological numerical models.

The functional form of this generalized vertical coordinate transformation, in terms of the original Cartesian system, can be written as

$$\begin{aligned} \tilde{x}^{1} &= x & \qquad x &= \tilde{x}^{1} \\ \tilde{x}^{2} &= y & \qquad y &= \tilde{x}^{2} \\ \tilde{x}^{3} &= \sigma(x,y,z,t) & \qquad z &= h(\tilde{x}^{1}, \tilde{x}^{2}, \tilde{x}^{3}, t) \end{aligned}$$

where σ can be given by (1).

The contravariant and covariant forms of the metric tensor $\tilde{G}^{ij}$ and $\tilde{G}_{ij}$ are given as

$$\tilde{G}^{ij} = \frac{\partial \tilde{x}^{i}}{\partial x^{l}} \frac{\partial \tilde{x}^{j}}{\partial x^{l}}$$

$$= \begin{bmatrix} 1 & 0 & \frac{\partial \sigma}{\partial x} \\ 0 & 1 & \frac{\partial \sigma}{\partial y} \\ \frac{\partial \sigma}{\partial x} & \frac{\partial \sigma}{\partial y} & \left(\frac{\partial \sigma}{\partial x}\right)^{2} + \left(\frac{\partial \sigma}{\partial y}\right)^{2} + \left(\frac{\partial \sigma}{\partial z}\right)^{2} \end{bmatrix}$$

$$\tilde{G}_{ij} = \frac{\partial x^l}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}$$

$$= \begin{vmatrix} 1+\left(\frac{\partial h}{\partial \tilde{x}^1}\right)^2 & \frac{dl\ h}{\partial \tilde{x}^1}\frac{dhh}{\partial \tilde{x}^2} & \frac{d}{\partial \tilde{x}^1}\frac{h}{\partial \tilde{x}^3} \\ \frac{\partial h}{\partial \tilde{x}^1}\frac{\partial h}{\partial \tilde{x}^2} & 1+\left(\frac{\partial h}{\partial \tilde{x}^2}\right)^2 & \frac{\partial h}{\partial \tilde{x}^2}\frac{\partial h}{\partial \tilde{x}^3} \\ \frac{\partial h}{\partial \tilde{x}^1}\frac{\partial h}{\partial \tilde{x}^2} & \frac{\partial h}{\partial \tilde{x}^2}\frac{\partial h}{\partial \tilde{x}^3} & \left(\frac{\partial h}{\partial \tilde{x}^3}\right)^2 \end{vmatrix}$$

while the only **nonzero** Christoffel symbol is

$$\tilde{\Gamma}^3_{jl} = \frac{\partial \sigma}{\partial z}\frac{\partial^2 h}{\partial \tilde{x}^j \partial \tilde{x}^l} ,$$

so that the covariant derivative of velocity is given by

$$\tilde{u}^i_{,j} = \begin{cases} \frac{\partial \tilde{u}^i}{\partial \tilde{x}^j} , & i = 1,2 \\ \frac{\partial \tilde{u}^3}{\partial \tilde{x}^j} + \tilde{\Gamma}^3_{jl}\tilde{u}^l , & i = 3 \end{cases}$$

determinant of the Jacobian of the transformation,

$$\left|\frac{\partial x^i}{\partial \tilde{x}j}\right| = \tilde{G}^{1/2}$$

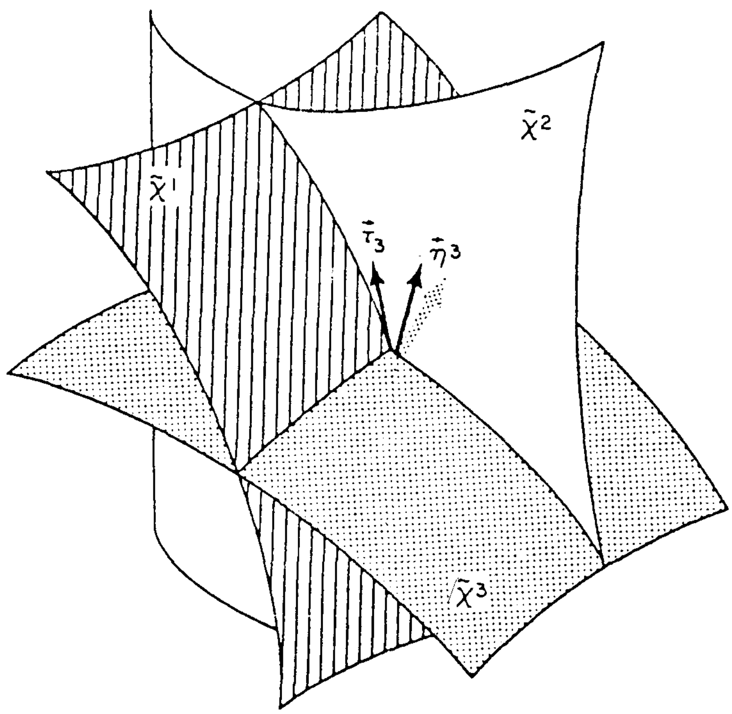
by

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial h}{\partial \tilde{x}^1} & \frac{\partial h}{\partial \tilde{x}^2} & \frac{\partial h}{\partial \tilde{x}^3} \end{vmatrix} = \tilde{G}^{1/2} = \frac{\partial h}{\partial \tilde{x}^3} = \frac{\partial h}{\partial \sigma}$$

The tangent and normal basis vectors for the generalized vertical coordinate system in terrm of the rectangular representation are given by

$$\left.\begin{aligned}
\tau_1 &= i + k\frac{\partial h}{\partial \tilde{x}^1}, & \eta^1 &= i,\\
\tau_2 &= j + k\frac{\partial h}{\partial \tilde{x}^2}, & \eta^2 &= j,\\
\tau_3 &= k\frac{\partial h}{\partial \tilde{x}^3} & \eta^3 &= i\frac{\partial \sigma}{\partial x} + j\frac{\partial \sigma}{\partial y} + k\frac{\partial \sigma}{\partial z}
\end{aligned}\right\} \quad (4)$$

where, since $\tau_i \cdot \tau_j$ does not equal zero when $i \neq j$, this coordinate system in general is nonorthogonal. in the original rectangular coordinate system, the normal and tangent basis functions are the same (i.e., i,j and k) and are orthogonal to one another.



The individual contravariant and covariant velocity components are found from $\tilde{u}^i = \eta^i \cdot u$ and $\tilde{u}_i = \tau_i \cdot u$, respectively, where $u = ui + vj + wk$, so that

$$\tilde{u}^1 = u, \qquad \tilde{u}_1 = u + \frac{\partial h}{\partial \tilde{x}^1} w$$

$$\tilde{u}^2 = v, \qquad \tilde{u}_2 = v + \frac{\partial h}{\partial \tilde{x}^2} w$$

$$\tilde{u}^3 = u\frac{\partial \sigma}{\partial x} + v\frac{\partial \sigma}{\partial y} + w\frac{\partial \sigma}{\partial z}, \qquad \tilde{u}_3 = w\frac{\partial h}{\partial \tilde{x}^3}.$$

Kinetic energy is computed from these expressions by

$$e = 1/2(\tilde{u}^1\tilde{u}^1 + \tilde{u}^2\tilde{u}^2 + \tilde{u}^3\tilde{u}^3).$$

As mentioned earlier, averaging of (3) is required if these equations are to be used in meteorological numerical models with finite grid and time intervals. The correct form of averaging this equation is called the grid-volume average and is given by

$$\overline{(\quad)} = \int_t^{t+\Delta t} \int_{\tilde{x}^1}^{\tilde{x}^1+\Delta\tilde{x}^1} \int_{\tilde{x}^2}^{\tilde{x}^2+\Delta\tilde{x}^2} \int_{\sigma}^{\sigma+\Delta\sigma} (\quad) d\sigma d\tilde{x}^2 d\tilde{x}^1 dt \,/\, (\Delta\tilde{x}^1)(\Delta\tilde{x}^2)(\Delta\sigma)(\Delta t) \qquad (5)$$

The dependent variables can be decomposed into average and s subgrid-scale perturbation expressed as

$$\phi = \overline{\phi} + \phi'',$$

where ϕ'' is a deviation from the subgrid-volume average. The symbol ϕ represents any one of the dependent variables.

Using (5), Eq. (3) can be rewritten as

$$\frac{\partial \overline{\tilde{u}}^i}{\partial t} = -\overline{\tilde{u}}^j \overline{\tilde{u}}^i_{,j} - \overline{\tilde{u}^{j''} \tilde{u}^{i''}_{,j}} - G^{ij}\overline{\theta}\frac{\partial \overline{\pi}}{\partial \tilde{x}^j} - \frac{\overline{\partial \tilde{x}^i}}{\partial z} g - 2\tilde{\varepsilon}^{ijl}\tilde{\Omega}_j \overline{\tilde{u}}_l. \qquad (6)$$

In deriving this form it has been assumed that $\theta = \overline{\theta}(1 \cdot F\, \theta'' \theta^{-1}) \cong \tilde{\theta}$ and that

$$\overline{\overline{\tilde{u}}}^i = \overline{\tilde{u}}^i; \qquad \frac{\partial\ u_i}{\partial\ t} = \frac{\partial\ \overline{u}_i}{\partial\ t}; \quad \text{etc. } (therefore\ \overline{\tilde{u}^{i''}} = 0, \quad etc.) \tag{7}$$

Assumption (7), when applied in a rectangular coordinate system, is called Reynold's averaging. To make this assumption in the transformed coordinate system, however, it is necessary to require that changes of the metric tensor over the four-dimensional grid-volume $\Delta\tilde{x}^1\Delta\tilde{x}^2\Delta\sigma\Delta t$ are small, since this tensor appears in (6). Expressed mathematically, this requirement can be written as

$$\overline{\tilde{G}^{ij}} = \int_t^{t+\Delta t} \int_{\tilde{x}^1}^{\tilde{x}^1+\Delta\tilde{x}^1} \int_{\tilde{x}^2}^{\tilde{x}^2+\Delta\tilde{x}^2} \int_{\sigma}^{\sigma+\Delta\sigma} (\tilde{G}^{ij})\, d\sigma\, d\tilde{x}^2 d\tilde{x}^1 dt \,/\, (\Delta\tilde{x}^1)(\Delta\tilde{x}^2)(\Delta\sigma)(\Delta t) \cong \tilde{G}^{ij}.$$

This requirement has significant implication on the choice of the vertical generalized coordinate since it must be selected such that variations of the gradient of the transformed coordinate within the grid volume are small compared with the grid volume averaged gradient.

The **advective** term in (6) is derived from

$$\overline{\tilde{u}^j \tilde{u}^i_{,j}} = \overline{\tilde{u}^j \frac{\partial\ \tilde{u}^i}{\partial\ \tilde{x}^j}} + \overline{\Gamma^i_{jl}\tilde{u}^j\tilde{u}^l} \cong \overline{\tilde{u}^j \frac{\partial\ \tilde{u}^i}{\partial\ x^j}} + \overline{\tilde{\Gamma}}^i_{jl}\overline{\tilde{u}^j\tilde{u}^l}$$

$$\cong \overline{\tilde{u}}^j \frac{\partial\ \overline{\tilde{u}}^i}{\partial\ \tilde{x}^j} + \overline{\tilde{u}^{j''} \frac{\partial\ \tilde{u}^{i''}}{\partial\ \tilde{x}^j}} + \overline{\tilde{\Gamma}}^i_{jl}[\overline{\tilde{u}}^j\overline{\tilde{u}}^l + \overline{\tilde{u}^{j''}\tilde{u}^{l''}}] = \overline{\tilde{u}}^j\overline{\tilde{u}}^i_{,j} + \overline{\tilde{u}^{j''}\tilde{u}^{i''}_{,j}},$$

where the assumption that changes of the metric tensor and its derivatives are small permits the removal of the Christoffel symbol from the integrand [this assumption can also be written as].

$$\tilde{\Gamma}^i_{jl} = \overline{\tilde{\Gamma}}^i_{jl} + \tilde{\Gamma}''^i_{jl} = \overline{\tilde{\Gamma}}^3_{jl} + \tilde{\Gamma}''^3_{jl} = \overline{\tilde{\Gamma}}^3_{jl}$$

where $\quad \left|\tilde{\Gamma}''^3_{jl}\right| < \left|\overline{\tilde{\Gamma}}^3_{jl}\right|$]

The Coriolis term can be expanded as

$$2\tilde{\varepsilon}^{ijl}\tilde{\Omega}_j\tilde{\bar{u}}_l = \frac{2\tilde{\Omega}_j\tilde{G}_{lm}\tilde{\bar{u}}^m\varepsilon_{ijl}}{\sqrt{\tilde{G}}} = 2\varepsilon_{ijl}\frac{\partial x^r}{\partial \tilde{x}^j}\Omega_r\tilde{G}_{lm}\tilde{\bar{u}}^m\frac{\partial\sigma}{\partial z}$$

with $\Omega_r = (0, \Omega\cos\phi, \Omega\sin\phi) = (0, \hat{f}/2, f/2)$.

In addition

$$\frac{\partial h}{\partial \tilde{x}^3}\frac{\partial\sigma}{\partial x^3} = 1$$

and

$$\frac{\partial \tilde{x}^1}{\partial z} = \frac{\partial \tilde{x}^2}{\partial z} = 0$$

with the decomposition of variables into resolvable and subgrid scale terms Eq(6) can, therefore, be written for the generalized vertical coordinate representation in component form as

$$\begin{aligned}\frac{\partial \tilde{\bar{u}}^1}{\partial t} = &-\tilde{\bar{u}}^j\frac{\partial \tilde{\bar{u}}^1}{\partial \tilde{x}^j} - \overline{\tilde{u}^{j''}\frac{\partial \tilde{u}^{1''}}{\partial \tilde{x}^j}} - \bar{\theta}\frac{\partial \bar{\pi}}{\partial \tilde{x}^1} - \bar{\theta}\frac{\partial\sigma}{\partial x}\frac{\partial \bar{\pi}}{\partial \tilde{x}^3} \\ &-\hat{f}\left(\frac{\partial h}{\partial \tilde{x}^1}\tilde{\bar{u}}^1 + \frac{\partial h}{\partial \tilde{x}^2}\tilde{\bar{u}}^2 + \frac{\partial h}{\partial \tilde{x}^3}\tilde{\bar{u}}^3\right) + f\tilde{\bar{u}}^2\end{aligned} \quad (8)$$

$$\frac{\partial \tilde{\bar{u}}^2}{\partial t} = -\tilde{\bar{u}}^j\frac{\partial \tilde{\bar{u}}^2}{\partial \tilde{x}^j} - \overline{\tilde{u}^{j''}\frac{\partial \tilde{u}^{2''}}{\partial \tilde{x}^j}} - \bar{\theta}\frac{\partial \bar{\pi}}{\partial \tilde{x}^2} - \bar{\theta}\frac{\partial\sigma}{\partial y}\frac{\partial \bar{\pi}}{\partial \tilde{x}^3} - f\tilde{\bar{u}}^1 \quad (9)$$

$$\begin{aligned}\frac{\partial \tilde{\bar{u}}^3}{\partial t} = &-\tilde{\bar{u}}^j\frac{\partial \tilde{\bar{u}}^3}{\partial \tilde{x}^j} - \overline{\tilde{u}^{j''}\frac{\partial \tilde{u}^{3''}}{\partial \tilde{x}^j}} - \bar{\tilde{\Gamma}}^3_{jl}\tilde{\bar{u}}^j\tilde{\bar{u}}^l - \bar{\tilde{\Gamma}}^3_{jl}\overline{\tilde{u}^{j''}\tilde{u}^{l''}} \\ &-\bar{\theta}\left\{\frac{\partial\sigma}{\partial x}\frac{\partial \bar{\pi}}{\partial \tilde{x}^1} + \frac{\partial\sigma}{\partial y}\frac{\partial \bar{\pi}}{\partial \tilde{x}^2} + \left[\left(\frac{\partial\sigma}{\partial x}\right)^2 + \left(\frac{\partial\sigma}{\partial y}\right)^2 + \left(\frac{\partial\sigma}{\partial z}\right)^2\right]\frac{\partial \bar{\pi}}{\partial \tilde{x}^3}\right\} \\ &+\left(\hat{f} + \frac{\partial h}{\partial \tilde{x}^2}f\right)\frac{\partial\sigma}{\partial z}\left[\left(1 + \left(\frac{\partial h}{\partial \tilde{x}^1}\right)^2\right)\tilde{\bar{u}}^1 + \frac{\partial h}{\partial \tilde{x}^1}\frac{\partial h}{\partial \tilde{x}^2}\tilde{\bar{u}}^2 + \frac{\partial h}{\partial \tilde{x}^1}\frac{\partial h}{\partial \tilde{x}^3}\tilde{\bar{u}}^3\right] \\ &-\frac{\partial h}{\partial \tilde{x}^1}f\frac{\partial\sigma}{\partial z}\left[\frac{\partial h}{\partial \tilde{x}^2}\frac{\partial h}{\partial \tilde{x}^1}\tilde{\bar{u}}^1 + \left(1 + \left(\frac{\partial h}{\partial \tilde{x}^2}\right)^2\right)\tilde{\bar{u}}^2 + \frac{\partial h}{\partial \tilde{x}^2}\frac{\partial h}{\partial \tilde{x}^3}\tilde{\bar{u}}^3\right] - \frac{\partial\sigma}{\partial z}g.\end{aligned} \quad (10)$$

The vector velocity $v = \tilde{\bar{u}}^j \tau_j, \tilde{\bar{u}}^3$ is in the same direction as the cartesian velocity $\bar{w}$, whereas $\tilde{\bar{u}}^1$ and $\tilde{\bar{u}}^2$ are, in general, at some angle to $\bar{u}$ and $\bar{v}$ in the original rectangular system as shown by **Eq** (4)

The Hydrostatic Assumption

To illustrate the effect of utilizing the hydrostatic assumption in **(8-10),** it is convenient to use (1) as the generalized vertical coordinate. The relation between the spatial coordinates in the two representations is given by

$$\tilde{x}^1 = x \qquad\qquad x = \tilde{x}^1$$

$$\tilde{x}^2 = Y \qquad\qquad Y = \tilde{x}^2$$

$$\tilde{x}^3 = \rho = s\frac{[z - z_G(x,y)]}{[s - z_G(x,y)]} \qquad\qquad z = h = \frac{\sigma}{s}\left[s - z_G(\tilde{x}^1, \tilde{x}^2)\right] + z_G(\tilde{x}^1, \tilde{x}^2)$$

so that the **nonzero** quantities needed to evaluate the Jacobian, metric tensor and Christoffel symbol are given as

$$\left.\begin{aligned}
\frac{\partial\sigma}{\partial x} &= \frac{\partial z_G}{\partial x}\left(\frac{\sigma - s}{s - z_G}\right), &\quad \frac{\partial h}{\partial \tilde{x}^1} &= \frac{\partial z_G}{\partial \tilde{x}^1}\left(\frac{s-\sigma}{s}\right) \\
\frac{\partial\sigma}{\partial y} &= \frac{\partial z_G}{\partial y}\left(\frac{\sigma - s}{s - z_G}\right), &\quad \frac{\partial h}{\partial \tilde{x}^2} &= \frac{\partial z_G}{\partial \tilde{x}^2}\left(\frac{s-\sigma}{s}\right) \\
\frac{\partial\sigma}{\partial z} &= \frac{s}{s - z_G}, &\quad \frac{\partial h}{\partial\sigma} &= \frac{s - z_G}{s}
\end{aligned}\right\} \tag{12}$$

and

$$\tilde{\Gamma}^3_{11} = \frac{s-\sigma}{s-z_G}\frac{\partial^2 z_G}{\partial \tilde{x}^{1^2}}; \quad \tilde{\Gamma}^3_{22} = \frac{s-\sigma}{s-z_G}\frac{\partial^2 z_G}{\partial \tilde{x}^{2^2}}; \quad \tilde{\Gamma}^3_{21} = \frac{s-\sigma}{s-z_G}\frac{\partial^2 z_G}{\partial \tilde{x}^1 \partial \tilde{x}^2}$$

$$\tilde{\Gamma}^3_{23} = -\frac{1}{s - z_G}\frac{\partial z_G}{\partial \tilde{x}^2}, \quad \tilde{\Gamma}^3_{13} = -\frac{1}{s - z_G}\frac{\partial z_G}{\partial \tilde{x}^1},$$

with

$$\tilde{\Gamma}^3_{21} = \tilde{\Gamma}^3_{12}, \quad \tilde{\Gamma}^3_{23} = \tilde{\Gamma}^3_{32}, \quad \tilde{\Gamma}^3_{13} = \tilde{\Gamma}^3_{31} \tag{13}$$

The individual contravariant components can be expressed in terms of the rectangular components as

$$
\begin{aligned}
\bar{\tilde{u}}^1 &= \bar{u}, \\
\bar{\tilde{u}}^2 &= \bar{v}, \\
\bar{\tilde{u}}^3 &= \bar{u}\frac{\partial z_G}{\partial x}\left(\frac{\sigma - s}{s - z_G}\right) + \bar{v}\frac{\partial z_G}{\partial y}\left(\frac{\sigma - s}{s - z_G}\right) + \bar{w}\frac{s}{s - z_G}.
\end{aligned}
$$

Using (12) and (13) in (8)-(10), it follows that this system of equations involves more nonzero terms than in the original specification in the rectangular system. The extra terms are particularly evident in the vertical equation in which Christoffel symbols appear, and in the Coriolis terms.

At this point it is appropriate to introduce the hydrostatic assumption. From (4), it is evident that

$$\eta^1 = i, \quad \eta^2 = j, \quad \eta^3 \approx k\frac{\partial \sigma}{\partial z}$$

when

$$\left|\frac{\partial \sigma}{\partial x}\right| << \left|\frac{\partial \sigma}{\partial z}\right| \quad , \quad \left|\frac{\partial \sigma}{\partial y}\right| << \left|\frac{\partial \sigma}{\partial z}\right| \tag{14}$$

which permits (10) to be rewritten as

$$
\begin{aligned}
\frac{\partial \bar{\tilde{u}}^3}{\partial t} = -\bar{\tilde{u}}^3 \frac{\partial \bar{\tilde{u}}^3}{\partial \tilde{x}^j} - \overline{\tilde{u}^{j''} \frac{\partial \bar{\tilde{u}}^{3''}}{\partial \tilde{x}^j}} - \Gamma^3_{jl} \bar{\tilde{u}}^j \bar{\tilde{u}}^l - \Gamma^3_{jl} \overline{\tilde{u}^{j''} \tilde{u}^{l''}} \\
- \bar{\theta}\left(\frac{\partial \sigma}{\partial z}\right)^2 \frac{\partial \bar{\pi}}{\partial \tilde{x}^3} - \frac{\partial \sigma}{\partial z} g - \hat{f}\frac{\partial \sigma}{\partial z}\bar{\tilde{u}}^1,
\end{aligned}
\tag{15}
$$

as long as the magnitude of $\partial \bar{\pi} / \partial \tilde{x}^3$ is at least as large as that of $\partial \bar{\pi} / \partial \tilde{x}^1$ and $\partial \bar{\pi} / \partial \tilde{x}^2$.

If the hydrostatic assumption is applied, where acceleration in the σ direction [which is essentially verrtical as given by (14)] and the Coriolis terms are much less than the pressure gradient and the gravitational acceleration terms, then (15) reduces to

$$\frac{\partial \bar{\pi}}{\partial \tilde{x}^3} = \frac{\partial \bar{\pi}}{\partial \sigma} = -\frac{g}{\theta}\left(\frac{\partial \sigma}{\partial z}\right)^{-1} = -\frac{g}{\theta}\frac{s - z_g}{s} \tag{16}$$

similary, (8) and (9) reduce to

$$\frac{\partial \bar{\tilde{u}}^1}{\partial t} = -\bar{\tilde{u}}^j \frac{\partial \bar{\tilde{u}}^1}{\partial \tilde{x}^j} - \overline{\tilde{u}^{j''} \frac{\partial \tilde{u}^{1''}}{\partial \tilde{x}^j}} - \bar{\theta}\frac{\partial \bar{\pi}}{\partial \tilde{x}^1} + g\frac{\sigma - s}{s}\frac{\partial z_G}{\partial x} - \hat{f}\bar{\tilde{u}}^3 + f\bar{\tilde{u}}^2 \tag{17}$$

$$\frac{\partial \bar{\tilde{u}}^2}{\partial t} = -\bar{\tilde{u}}^j \frac{\partial \bar{\tilde{u}}^2}{\partial \tilde{x}^j} - \overline{\tilde{u}^{j''} \frac{\partial \tilde{u}^{2''}}{\partial \tilde{x}^j}} - \bar{\theta}\frac{\partial \bar{\pi}}{\partial \tilde{x}^2} + g\frac{\sigma - s}{s}\frac{\partial z_G}{\partial y} + f\bar{\tilde{u}}^1 \tag{18}$$

Eq (16)-(18) are in the form obtained when the chain rule is applied separately for the horizontal and the vertical equations of motion. As shown here, however, (16)-(18) are only approximate relationships when a complete tensor transformation is applied, and they are valid only when, (14) applies. The terms given in (14) can also be written as

$$\left|\frac{\partial \sigma}{\partial x}\right|_{\max} = \left|\frac{\partial z_G}{\partial x}\right|, \quad \left|\frac{\partial \sigma}{\partial y}\right|_{\max} = \left|\frac{\partial z_G}{\partial y}\right|.$$

Hence, the inequality given by (14) states that $|\partial z_G / \partial x| \approx |\partial z_G / \partial y| << 1$ is a necessary condition to assure the validity of (16)-(18). In terms of the terrain representation, this condition requires that the slope must have an angle << 45”.

The subgrid-scale terms which are included in (17) and (18), must also be parameterized in terms of known quantities in order to completely specify these equation. In the original rectangular coordinate system, it is the customary practice to decompose the subgrid-scale terms into vertical and horizontal components, such that, for the equation of motion with I = 1, for example,

$$-\overline{\rho} w'' u'' = F_{z_u}, \quad -\overline{\rho} u^{j''} u'' = F^J_{H_u} \quad (j = 1,2)$$

where F_{z_u} represents the vertical turbulent fluxes of the east-west, u, component of velocity, while $F^J_{H_u}$ indicate the horizontal turbulent fluxess of u. This separation into two components in mesoscale models is necessitated for two major reasons:

1. In most mesoscale models, the horizontal grid spacing $(\Delta x, \Delta y)$ is much larger than the vertical spacing (Az) so that the parameterization of subgrid scale mixing in the horizontal and vertical directions would be expected to be quite different.

2. Much more is known about the functional form of vertical subgrid scale fluxes than of horizontal subgrid-scale fluxes. Thus, two completely different parameterizations are required, with the vertical flux representation being much more detailed.

In a terrain-following coordinate system, when

$$\left|\frac{\partial z_G}{\partial x}\right| \approx \left|\frac{\partial z_G}{\partial y}\right| << 1,$$

it, therefore, is desirable to retain this separation into vertical and horizontal flux components, To illustrate this, multiply the first two terms on the right-hand side of the equality in (17) by $\overline{\rho}(s - z_G)/s$ so that

The transformed grid-volume average conservation of mass relation, can be written as

$$\overline{\rho}\frac{(s - z_G)}{s}\left[\tilde{\overline{u}}^j \frac{\partial \tilde{\overline{u}}^1}{\partial \tilde{x}^j} + \overline{\tilde{u}^{i''} \frac{\partial u^{1''}}{\partial \tilde{x}^j}}\right]$$

$$= \overline{\rho}\frac{(s - z_G)}{s}\left[\overline{\left(\tilde{\overline{u}}^j + \tilde{u}^{j''}\right)\frac{\partial}{\partial \tilde{x}^j}\left(\tilde{\overline{u}}^1 + \tilde{u}^{1''}\right)}\right]$$

$$= \frac{\partial}{\partial \tilde{x}^j}\overline{\rho}\frac{(s - z_G)}{s}\overline{(\tilde{\overline{u}}^j + \tilde{u}^{j''})(\tilde{\overline{u}}^1 + \tilde{u}^{1''})}$$

$$= \frac{\partial}{\partial \tilde{x}^j}\left[\overline{\rho}\frac{(s - z_G)}{s}\tilde{\overline{u}}^j\tilde{\overline{u}}^1\right] + \frac{\partial}{\partial \tilde{x}^j}\left[\overline{\rho}\frac{(s - z_G)}{s}\overline{\tilde{u}^{j''}\tilde{u}^{1''}}\right]$$

In writing expressing, me anelastic form of the conservation of mass equation in the transformed system, given by

$$\frac{1}{s-z_G}\frac{\partial}{\partial \tilde{x}^j}\overline{\rho}(s-z_G)\tilde{\bar{u}}^j = 0,$$

along with the assumption that $\overline{u^{j''}} = \overline{u^{1''}} = 0$ as defined by (7), has been used. Similar terms, of course, can be derived for the advection terms in (18).

Thus, in the transformed coordinate system the subgrid-scale fluxes are given as

$$\left.\begin{aligned}\overline{\rho}\frac{(s-z_G)}{s}\overline{\tilde{u}^{3''}\tilde{u}^{1''}} &= F_{\sigma_{\bar{u}^1}} \\ \overline{\rho}\frac{(s-z_G)}{s}\overline{\tilde{u}j''\tilde{u}^{1''}} &= F^j_{H_{u^{-1}}}\end{aligned}\right\} \quad (\text{j} = 1{,}2),$$

where $F_{\sigma_{\bar{u}^1}}$ and $F^J_{H_{\bar{u}^1}}$ are, respectively, the $\tilde{u}^1$ fluxes in the $\tilde{x}^3$ direction and in the $\tilde{x}^1$ ***and*** $\tilde{x}^2$ directions.

Since it is assumed that

$$\left|\frac{\partial\sigma}{\partial x^1}\right| \approx \left|\frac{\partial\sigma}{\partial x^2}\right| << 1$$

and so

$$\frac{\partial\sigma}{\partial z} = 1,$$

it is reasonable to also assume that the fluxes in the $\tilde{x}^3$ *and* x^3 directions in the two systems are almost equal and so

$$F_{z_u} \approx F_{\sigma_{\tilde{u}^1}} \approx \overline{\rho}\,\overline{w''u''}^{R} = \overline{\rho}\,\frac{s-z_G}{s}\,\overline{\tilde{u}^{3''}\tilde{u}^{1''}},$$

$$\overline{\tilde{u}^{3''}\tilde{u}^{1''}} \quad \overline{w''u''}^{R} \frac{\quad}{s-z_G} \quad \overline{\quad}^{R},$$

where the overbar with the R superscript is used to emphasize that this averageing volume is different from the given by (5) (in this case a rectangular volume). Moreover, if $\overline{w''u''}^{R}$ is assumed proportional to an exchange coefficient which is a function of height ξ above the ground and the mean velocity profile $\bar{u}^{R}$,as is often done, then

$$\overline{\tilde{u}^{3''}\tilde{u}^{1''}} \cong \frac{s}{s-z_G}\,w''u'' = -\frac{s}{s-z_G}\,K(\xi)\,\frac{\partial\,\bar{u}^{R}}{\partial\,z}$$

since $\tilde{u}^1 = \bar{u}, \xi = \sigma(s-z_G)/s$ and $\partial/\partial z \approx [s/(s-z_G)](\partial/\partial\,\tilde{x}^3)$, then this approximation for the vertical sub-grid scale flux becomes

$$\overline{\tilde{u}^{3''}\tilde{u}^{1''}} \cong -\left(\frac{s}{s-z_G}\right)^2 K_M\left(\sigma\frac{s-z_G}{s}\right)\frac{\partial\,\bar{\tilde{u}}^1}{\partial\,\tilde{x}^3}$$

and so the $\tilde{x}^3$ flux term in (7) can be represented as

$$\overline{\tilde{u}^{3''}\frac{\partial\,\tilde{u}^{1''}}{\partial\,\tilde{x}^3}} = \left(\frac{s}{s-z_G}\right)^2 \frac{\partial}{\partial\,\tilde{x}^3} K_M \frac{\partial\,\bar{\tilde{u}}^1}{\partial\,\tilde{x}^3}$$

where K is the function of $\sigma(s-z_G)/s$ (i.e., is a function of height above the ground). The subgrid flux in the $\tilde{x}^3$ direction in (18) can be shown to have the same form.

The subgrid-scale fluxes in the $\tilde{x}^1$ *and* $\tilde{x}^2$ directions could be written in a similar form; however, since essentially nothing is known about their function form on the mesoscale in the rectangular coordinate representation, no purpose is served by writting them here. Subgrid-scale fluxes in the horizontal direction are included in models for computation reasons only.

Conclusion

This paper uses tensor transformation procedures in order to derive a terain-following cooordinate system which is frequently used in a number of regional and mesoscale hydrostatic models. The technique utilizes tensor transformation procedures in order to ensure the physical invariance of the conservation relations between the Cartesian and terrain-following systems.

The analysis has shown that, in general, applying the chain rule separately to the hydrostatic equation and the horizontal equations of motion in order to transform them to a generalized vertical coordinate system yields a different form of equation than when the tensor transformation is applied before the hydrostatic assumption is made. Only when the slope of the terrain is much less than 45°, the two procedures of obtaining transformed equations will yield the sames form.

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