การหาค่าการนำความร้อน ที่ขึ้นกับอุณหภูมิในระบบการนำความร้อนสองมิติ

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บทคัดย่อ

บทความนี้นำเสนอวิธีเชิงตัวเลขสำหรับการคำนวณหาค่าการนำความร้อนที่ขึ้นกับอุณหภูมิของของแข็ง ปัญหาที่ใช้วิธีนี้แก้ได้คือปัญหาการนำความร้อนสองมิติคงตัว โดยวิธีดังกล่าวอาศัยการทำแปลงปัญหาไม่เชิงเส้นให้เป็นปัญหาเชิงเส้นด้วย Kirchhoff's transformation และระเบียบวิธี boundary element method ในการแก้ปัญหาเชิงเส้นที่ได้ การทดสอบวิธีที่นำเสนอนี้กระทำโดยใช้รูปแบบการทดลองตัวอย่างและทำทดลองจำลอง ข้อมูลจากการทดลองที่จำเป็นสำหรับการหาค่าการนำความร้อนได้มาจากการวัดอุณหภูมิและค่าความร้อนโดยรวมที่ให้กับของแข็ง ผลที่ได้จากการทดสอบแสดงให้เห็นว่าวิธีที่นำเสนอสามารถแก้ปัญหาได้อย่างมีประสิทธิผล

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Determination of Temperature-dependent Thermal Conductivity in a Two-dimensional Heat Conduction System

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Abstract

This article presents a numerical method for the determination of temperature-dependent thermal conductivity of a solid. Problems in which this method is intended for are steady-state two-dimensional heat conduction problems. The numerical method is based on the boundary element method, and requires the linearization of the the steady-state nonlinear heat equation by the Kirchhoff's transformation. Numerical tests of the method are performed by conducting simulated experiments using a sample experiment setup. Knowledge of total applied heat input and steady-state temperature measurements is required. The results of the tests show that the proposed method can effectively provide an accurate solution.

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1. Introduction

Because thermal conductivity is an important property needed in thermal design and analysis, there is an ongoing interest in determining thermal conducitivities of new materials. Conventional experimental methods for measuring the thermal conductivity are based on the assumptions that heat conduction can be described by the Fourier's law, and that heat flow is unidirectional [1]. Although the first assumption is generally valid for most materials of interest, the second assumption usually requires careful experimental design to avoid multidimensional heat flow that will contribute to errors. Obviously, experimental design will be more flexible if multidimensional heat flow is allowed. In such an experiment, the boundary condition is known, and some temperature measurements inside the material or on the boundary are available. The resulting problem is known as an inverse problem, which can be solved for unknown thermal conductivity. The inverse problem of determining thermal conductivity has been given considerable interest. Previous studied problems include problems of determining thermal conductivity and heat capacity simultaneously [2]-[6] and problems of determining only thermal conductivity [7]-[9]. Most of these problems are one-dimensional, transient problems [2], [3], [5]-[9]. Multidimensional problems have hitherto received considerably less interest despite their greater ability to model actual experiments.

In this paper, consideration is given to the two-dimensional problem of determining temperature-dependent thermal conductivity. The heat conduction model used will be a steady-state model. The numerical method that can be efficiently used to solve the resulting problem is the boundary element method. In the following sections, the statement of the problem will be described. The boundary element formulation for solving the inverse problem will then be presented. Then proposed method will be used to solve a sample problem, which models a sample experimental setup. Presentation and discussion of the results will follow. Finally, the paper will conclude with suggestions for future works in this area.

2. Statement of the Problem

Consider a two-dimensional homogeneous solid, of which thermal conductivity is unknown. The boundary of the solid is divided into three parts. The first part (Γ_1) is subjected to heat input. The distribution of the heat input along Γ_1 is unknown, but the total heat input is known. The second part (Γ_2) is insulated. The third part (Γ_3) is in contact with a thermal reservoir. Hence, temperature along Γ_3 is kept constant.

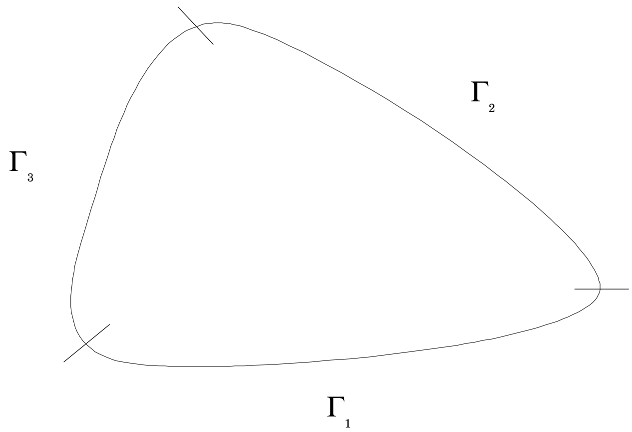

Fig. 1 Heat conduction in a homogeneous two-dimensional solid

The governing equation and the boundary condition for the model shown in Fig. 1 are

$$\vec{\nabla}\left(k(T)\vec{\nabla}T(\vec{r})\right) \quad = \quad 0 \tag{1}$$

$$-k(T)\vec{n}\vec{\nabla}T\Big|_{\Gamma_1} \quad = \quad \dot{q}^{(1)}(\vec{r})\Big|_{\vec{r}\text{ on }\Gamma_1} \tag{2}$$

$$\vec{n}\vec{\nabla}T\Big|_{\Gamma_2} \quad = \quad 0 \tag{3}$$

$$T(\vec{r})\Big|_{\vec{r}\text{ on }\Gamma_3} \quad = \quad T_r \tag{4}$$

where $\vec{n}$ denotes the unit vector normal to and pointing away from boundary, $\dot{q}^{(1)}$ is heat flux on Γ_1, and T_r is the temperature of the reservoir. Although heat flux distribution on Γ_1 is not known, the total heat input $\dot{Q}$ is assumed to be given.

$$\dot{Q} = \int_{\Gamma_1} \dot{q}(\vec{r})\mathrm{d}\Gamma \tag{5}$$

Additional information that will be available for solving the above equations is temperature measurements on Γ_1 and Γ_2.

3. Boundary Element Formulation

Eqs. (1)-(4) are a nonlinear system, which can be linearized by the Kirchhoff's transformation.

$$\theta \quad = \int_{T_r}^{T} k(\xi)\mathrm{d}\xi \tag{6}$$

The resulting linearized system is

$$\nabla^2\theta(\vec{r}) = 0 \quad (7)$$

$$-\vec{n}\vec{\nabla}\theta(\vec{r})\Big|_{\vec{r}\text{ on }\Gamma_1} = \dot{q}^{(1)}(\vec{r})\Big|_{\vec{r}\text{ on }\Gamma_1} \quad (8)$$

$$\vec{n}\vec{\nabla}\theta(\vec{r})\Big|_{\vec{r}\text{ on }\Gamma_2} = 0 \quad (9)$$

$$\theta(\vec{r})\Big|_{\vec{r}\text{ on }\Gamma_3} = \theta_r \quad (10)$$

The boundary element formulation for above system is given by [10]

$$a_k\theta(\vec{r}_k) = \int_\Gamma \dot{q}(\vec{r})G(\vec{r}-\vec{r}_k)d\Gamma - \int_\Gamma \theta(\vec{r})\,\vec{n}\vec{\nabla}G(\vec{r}-\vec{r}_k)d\Gamma \quad (11)$$

where a_k is coefficient that depends on position vector $\vec{r}_k$, and the fundamental solution G is

$$G(\vec{r}-\vec{r}_k) = \frac{1}{2\pi}\ln\left(\frac{1}{|\vec{r}-\vec{r}_k|}\right) \quad (12)$$

Divide the boundary into N quadratic boundary elements, and make use of quadratic interpolating functions to approximate variations of θ and $\dot{q}$ along elements. The result is a system of linear equations relating $\theta(\vec{r}_k)$ to θ_j and $\dot{q}_j$ at boundary nodes.

$$a_k\theta(\vec{r}_k) = \sum_{j=1}^{M}\psi(\vec{r}_j-\vec{r}_k)\dot{q}_j - \sum_{j=1}^{M}\phi(\vec{r}_j-\vec{r}_k)\theta_j \quad (13)$$

where M is the number of boundary nodes, and functions ϕ and ψ are obtained from evaluations of integrals in Eq. (11). Eq. (13) can now be written for all boundary nodes. The result is a system of M linear equations. It can be solved for heat flux components $\dot{q}_j^{(1)}$ on Γ_1 and $\dot{q}_j^{(2)}$ on Γ_2 in terms of $\theta_k^{(1)}$, $\theta_k^{(2)}$, and $\theta_k^{(3)}$, which are temperatures on Γ_1, Γ_2, and Γ_3, respectively.

$$\dot{q}_j^{(1)} = \sum_{k=1}^{M_1}C_{jk}^{(11)}\theta_k^{(1)} + \sum_{k=1}^{M_2}C_{jk}^{(12)}\theta_k^{(2)} + \sum_{k=1}^{M_3}C_{jk}^{(13)}\theta_k^{(3)} \quad (\text{for } j = 1, 2, \ldots, M_1) \quad (14)$$

$$\dot{q}_j^{(2)} = \sum_{k=1}^{M_1}C_{jk}^{(21)}\theta_k^{(1)} + \sum_{k=1}^{M_2}C_{jk}^{(22)}\theta_k^{(2)} + \sum_{k=1}^{M_3}C_{jk}^{(23)}\theta_k^{(3)} \quad (\text{for } j = 1, 2, \ldots, M_2) \quad (15)$$

where M_1, M_2, and M_3 are the numbers of boundary nodes on Γ_1, Γ_2, and Γ_3, respectively.

Now assume that thermal conductivity is a polynomial function of temperature.

$$k(T) = c_0 + c_1(T - T_r) + c_2(T - T_r)^2 + \ldots + c_n(T - T_r)^n \tag{16}$$

Evaluation of the intergral in Eq. (6) yields

$$\theta = c_0(T - T_r) + \frac{1}{2}c_1(T - T_r)^2 + \ldots + \frac{1}{n+1}c_n(T - T_r)^{n+1} \tag{17}$$

Substitute θ into Eq. (15), and set $\dot{q}_j^{(2)} = 0$.

$$\begin{aligned} 0 = {} & \sum_{k=1}^{M_1} C_{jk}^{(21)}\left[c_0\left(T_k^{(1)} - T_r\right) + \frac{1}{2}c_1\left(T_k^{(1)} - T_r\right)^2 + \ldots + \frac{1}{n+1}c_n\left(T_k^{(1)} - T_r\right)^{n+1}\right] + \\ & \sum_{k=1}^{M_2} C_{jk}^{(22)}\left[c_0\left(T_k^{(2)} - T_r\right) + \frac{1}{2}c_1\left(T_k^{(2)} - T_r\right)^2 + \ldots + \frac{1}{n+1}c_n\left(T_k^{(2)} - T_r\right)^{n+1}\right] \end{aligned} \tag{18}$$

Rearrange Eq. (18) so that there are n unknowns.

$$D_{j1}\frac{c_1}{2c_0} + D_{j2}\frac{c_2}{3c_0} + \ldots + D_{jn}\frac{c_n}{(n+1)c_0} = E_j \tag{19}$$

where

$$D_{ji} = \sum_{k=1}^{M_1} C_{jk}^{(21)}\left(T_k^{(1)} - T_r\right)^{i+1} + \sum_{k=1}^{M_2} C_{jk}^{(22)}\left(T_k^{(2)} - T_r\right)^{i+1} \tag{20}$$

$$E_j = \sum_{k=1}^{M_1} C_{jk}^{(21)}\left(T_k^{(1)} - T_r\right) + \sum_{k=1}^{M_2} C_{jk}^{(22)}\left(T_k^{(2)} - T_r\right) \tag{21}$$

In order to solve Eq. (19) for n unknowns, the number of equations M_2 must be greater than n.

For the determination of c_0, Eqs. (7) and (14) must be used. First, the integral in Eq. (5) is approximated numerically.

$$\dot{Q} = \sum_{j=1}^{M_1} D_j \dot{q}_j^{(1)} \tag{22}$$

where D_j is coefficient that determines the accuracy of the numerical integration. Next, substitute θ from Eq. (17) in Eq. (14). The resulting expression for $\dot{q}_j^{(1)}$ is then inserted into Eq. (22), yielding the following equation.

$$\dot{Q} = c_0\left\{\sum_{k=1}^{M_1} C_{jk}^{(11)}\left[\left(T_k^{(1)}-T_r\right)+\frac{c_1}{2c_0}\left(T_k^{(1)}-T_r\right)^2+\ldots+\frac{c_n}{(n+1)c_0}\left(T_k^{(1)}-T_r\right)^{n+1}\right]+\right.$$

$$\left.\sum_{k=1}^{M_2} C_{jk}^{(12)}\left[\left(T_k^{(2)}-T_r\right)+\frac{c_1}{2c_0}\left(T_k^{(2)}-T_r\right)^2+\ldots+\frac{c_n}{(n+1)c_0}\left(T_k^{(2)}-T_r\right)^{n+1}\right]\right\} \qquad (23)$$

Since c_1/c_0, c_2/c_0, ..., c_n/c_0 have been determined from Eq. (19), Eq. (23) can be solved for c_0.

4. Sample Experimental Setup

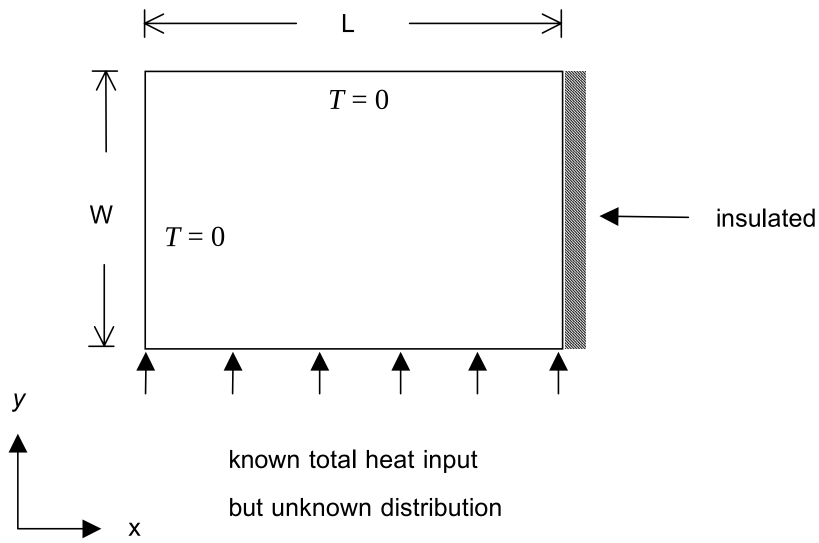

Fig. 2 Schematic of sample experimental setup for determining thermal conductivity

Several experimental designs based on the model described in the previous section are possible. One of the simplest experimental designs is shown in Fig. 2. A two-dimensional homogeneous solid of length L and width W is subjected to known total heat input at the bottom side. However, the knowledge of heat flux distribution is not required. The right side is insulated, and the top and left sides are in contact with a thermal reservoir having the temperature of 0 °C. Temperature measurements are taken from 19 sensors placed on the bottom and right sides. Sensors # 1 - 10 are located at (x, y) = (0.1L, 0), (0.2L, 0), ..., (L, 0). Sensors # 11 - 19 are located at (x, y) = (L, 0.1W), (L, 0.2W), ..., (L, 0.9W).

To demonstrate the effectiveness of the proposed method of determining thermal conductivity, heat flux $\dot{q}^{(1)}(x,0)$ applied to the bottom side is assumed to be

$$\dot{q}^{(1)}(x,0) = \frac{\pi\dot{Q}}{2L}\sin\left(\frac{\pi x}{2L}\right) \tag{24}$$

For this particular heat flux function, the exact solution for Eqs. (7) - (10) is

$$\theta(x, y) = \dot{Q}\sin\left(\frac{\pi x}{2L}\right)\left[\frac{\sinh\frac{\pi}{2L}(W-y)}{\cosh\frac{\pi W}{2L}}\right] \tag{25}$$

With a given thermal conductivity function $k(T)$, exact temperatures at 19 sensors location can be calculated. They are then used as input to Eqs. (19) and (23), which are solved for the unknown coefficients, c_0, c_1, ..., and c_n. Comparison between the computed coefficients and the coefficients in exact $k(T)$ will provide an assessment of effectiveness of the proposed method.

5. Results and Discussion

In order to test the accuracy of the numerical method, the square domain with W = L = 0.2 m was used, each of the four sides was divided into five quadratic elements, and exact temperatures were specified at the boundary nodes. Calculated total heat input at the bottom side was then compared with exact total heat input $\dot{Q}$. It was found that the L_2 norm of the difference between the numerical solution and exact solution was only around 0.08% of $\dot{Q}$. Furthermore, the L_2 norm of this difference monotonically decreased as more elements were used. Therefore, computed coefficients $C_{jk}^{(11)}$, $C_{jk}^{(12)}$, $C_{jk}^{(21)}$, and $C_{jk}^{(22)}$ are expected to closely approximate the exact values.

Numerical tests of the method will now be performed using the same computational details as those used in checking the accuracy of the method. In addition, the numerical integration in Eq. (22) is given by

$$\dot{Q} = 0.2L\left[\frac{1}{6}\left(\dot{q}_1^{(1)} + \dot{q}_{11}^{(1)}\right) + \frac{2}{3}\left(\dot{q}_2^{(1)} + \dot{q}_4^{(1)} + \dot{q}_6^{(1)} + \dot{q}_8^{(1)} + \dot{q}_{10}^{(1)}\right) + \frac{1}{3}\left(\dot{q}_3^{(1)} + \dot{q}_5^{(1)} + \dot{q}_7^{(1)} + \dot{q}_9^{(1)}\right)\right] \tag{26}$$

which results from the assumption of quadratic variation of heat flux along each boundary element. The thermal conductivity function used in the numerical tests is

$$k(T) = 100 - 2T + 0.02T^2 - 0.00004T^3 \tag{27}$$

The total heat input $\dot{Q}$ is the variable in the numerical tests. Three values of $\dot{Q}$ (20, 30, and 40 kW/m) are used to investigate the influence of $\dot{Q}$ on the solution. For each value, a total of five simulated experiments are performed. In each experiment, exact temperatures (T_{exa}) at 19 sensor locations are calculated. Fig. 3 shows that as $\dot{Q}$ is increased, temperatures will also increase, as expected.

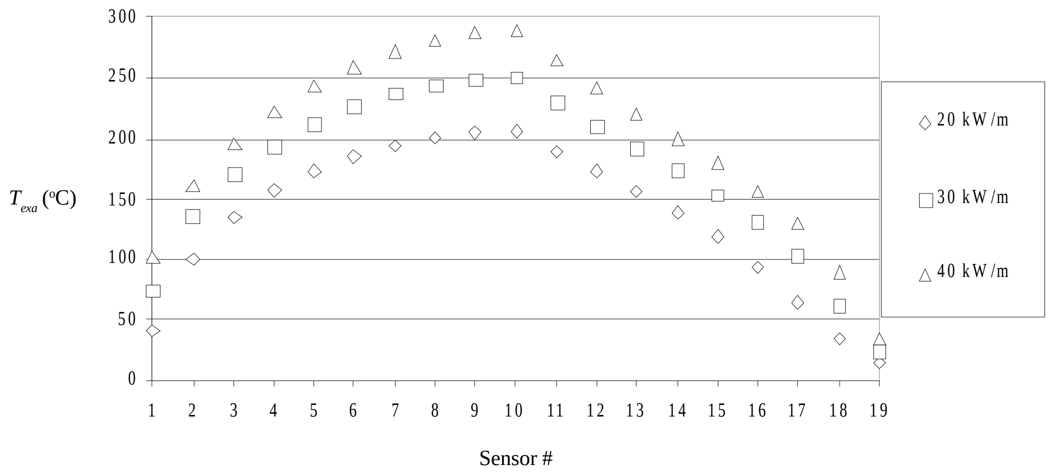

Fig. 3 Exact temperatures at 19 sensors locations for $\dot{Q}$ = 20, 30, and 40 kW/m

Simulated temperature measurements (T_{sim}) are obtained by adding statistical errors (ε) to exact temperatures.

$$T_{sim} = T_{exa} + \varepsilon \tag{28}$$

The error distribution is assumed to be Gaussian, with standard deviation σ = 0.1 °C. In addition, data acquisition system is assumed to display temperature readings to the first decimal point. Hence, simulated temperatures are rounded to the first decimal point. They are used as input to the algorithm for determining thermal conductivity. The thermal conductivity is reproduced five times in five simulated experiments for each value of $\dot{Q}$. The reproduced value in each simulated experiment is expected to be different because of the randomness of the statistical errors, which are generated using the RANLIB package (obtained from http://www.netlib.org/random/index.html).

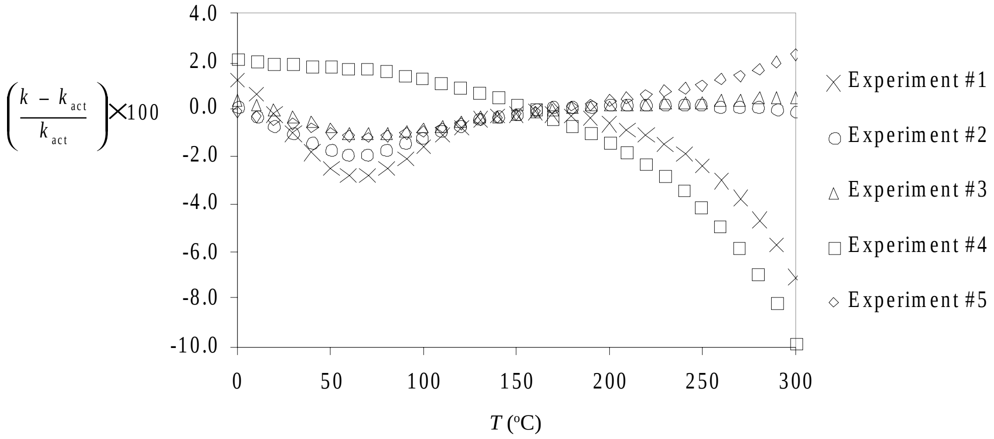

Fig. 4 Differences between k and k_{act} for $\dot{Q}$ = 20 kW/m

Table 1 Comparison of coefficients of thermal conductivity function for $\dot{Q}$ = 20 kW/m

	c_0	c_1	c_2	c_3
Actual	100	-2	0.02	-4.00×10^{-5}
Experiment #1	101.2703	-2.08722	0.020926	-4.27×10^{-5}
Experiment #2	99.9968	-2.02702	0.020257	-4.06×10^{-5}
Experiment #3	100.3312	-2.02064	0.020149	-4.02×10^{-5}
Experiment #4	102.0312	-2.04784	0.020581	-4.23×10^{-5}
Experiment #5	99.83412	-2.00573	0.01999	-3.97×10^{-5}

Figure 4 and Table 1 show the simulated experimental results for $\dot{Q}$ = 20 kW/m. It can be seen that estimated k differs from actual k by less than 10% in the temperature range 0-300 °C. Furthermore, estimated coefficients are reasonably close to actual coefficients.

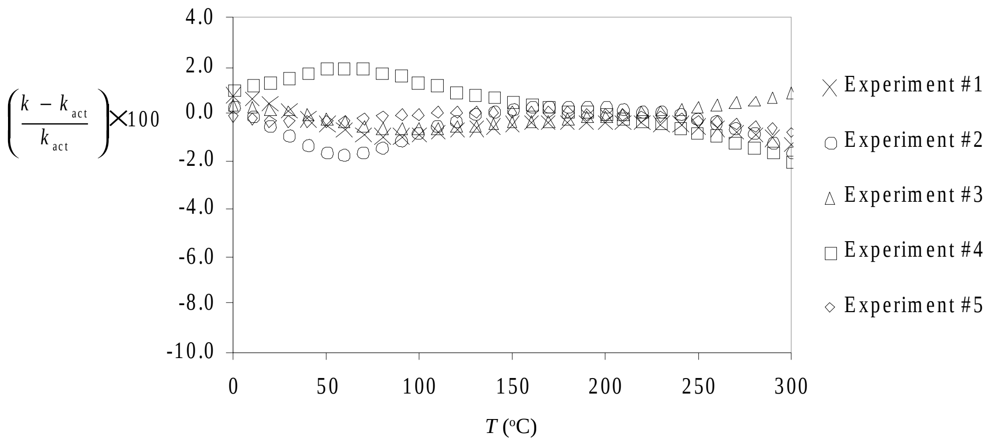

Fig. 5 Differences between k and k_{act} for $\dot{Q}$ = 30 kW/m

Table 2 Comparison of coefficients of thermal conductivity function for $\dot{Q}$ = 30 kW/m

	c_0	c_1	c_2	c_3
Actual	100	-2	0.02	-4.00×10^{-5}
Experiment #1	100.7797	-2.02964	0.020222	-4.05×10^{-5}
Experiment #2	100.2433	-2.03794	0.020418	-4.11×10^{-5}
Experiment #3	100.3259	-2.00856	0.02	-3.98×10^{-5}
Experiment #4	101.0022	-2.0068	0.020092	-4.04×10^{-5}
Experiment #5	99.84323	-2.00167	0.020053	-4.02×10^{-5}

Fig. 5 and Table 2 show the simulated experimental results for $\dot{Q}$ = 30 kW/m. The difference between estimated and actual thermal values in this case is less than in the previous case. Therefore, increasing the total applied heat input is expected to produce more accurate estimate of thermal conductivity. To see the reason for this, let's consider Eq. (23). If this equation is scaled by $\dot{Q}$, it can be seen that a larger value of $\dot{Q}$ reduces the effect of statistical variation in measured temperatures $T_k^{(1)}$ and $T_k^{(2)}$, and results in a more accurate estimate of c_0.

Fig. 6 and Table 3 show the simulated experimental results for $\dot{Q}$ = 40 kW/m. The difference between estimated and actual thermal values in this case is even less than in the previous two cases. This confirms the conclusion that the greater the applied heat input, the more accurate the solution. According to Fig. 3, temperatures at 19 sensors are largest when $\dot{Q}$ = 40 kW/m, and smallest when 20 kW/m. This implies that using high temperature measurements will make the determination of unknown thermal conductivity more accurate.

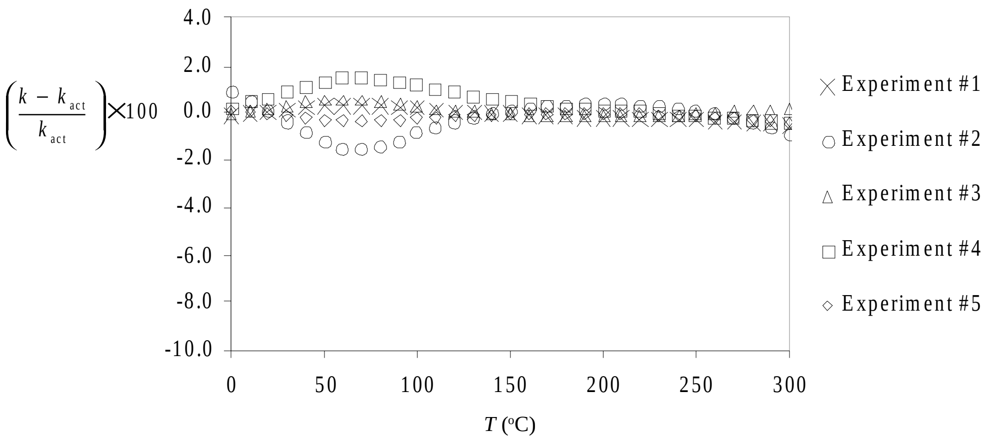

Fig. 6 Differences between k and k_{act} for $\dot{Q}$ = 40 kW/m

Table 3 Comparison of coefficients of thermal conductivity function for $\dot{Q}$ = 40 kW/m

	c_0	c_1	c_2	c_3
Actual	100	-2	0.02	-4.00×10^{-5}
Experiment #1	99.90753	-1.99317	0.019943	-3.99×10^{-5}
Experiment #2	100.8325	-2.04726	0.020441	-4.10×10^{-5}
Experiment #3	99.95339	-1.98934	0.019883	-3.97×10^{-5}
Experiment #4	100.2248	-1.9891	0.019931	-3.99×10^{-5}
Experiment #5	100.1006	-2.00766	0.020072	-4.02×10^{-5}

6. Conclusions

A numerical method for determining thermal conductivity of a solid is proposed. This method can deal with a heat conduction model in which heat flow is two-dimensional. A sample experimental setup with such a model is proposed for experimental measurement of temperature-dependent thermal conductivity. Unlike conventional setups, this setup should be more easily implemented since it does not rely on the one-dimensionality of heat flow, which will require that the applied heat input be uniform. Inputs to the numerical method include temperature measurements along surface of the solid, where sensors are readily accessible, and knowledge of total applied heat input, but not the heat flux distribution. Numerical tests of the method show its effectiveness in estimating unknown thermal conductivity.

This experimental setup is just one possible way to take advantage of the ability to solve virtually any inverse problem numerically in designing an experiment that suffers from fewer constraints than conventional experiments. However, the proposed setup is somewhat idealized in that it is two-dimensional. Future works in this area should involve designs of three-dimensional experimental setups, actual experiments, and numerical studies of the associated three-dimensional inverse problems.

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