การแก้สมการบาวดารีเลเยอร์แบบลามินาร์ อัดตัวไม่ได้ ด้วยรูปแบบไฟไนดิเฟอร์เรน

วันชัย อัศวภูษิตกุล [1]
มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าธนบุรี บางมด ทุ่งครุ กรุงเทพฯ 10140

บทคัดย่อ

รูปแบบ Time-accurate ได้ถูกพัฒนาเพื่อใช้ในการคำนวณ สมการบาวดารีเลเยอร์ แบบสองมิติลามินาร์ อัดตัวไม่ได้ โดยการหาคำตอบในช่วงเวลาที่เปลี่ยนไปด้วยวิธี Time stepping การแก้สมการโมเมนตัมจะใช้แบบ Central differencing ในทิศที่ตั้งฉากกับรูปร่างวัตถุและแบบ Upwind differencing ในทิศที่เหลือ รูปแบบสมการที่ต้องแก้หาคำตอบอยู่ในรูป Tridiagonal matrix วัตถุประสงค์หลักในการศึกษานี้เพื่อพัฒนากระบวนการคำนวณที่จะให้คำตอบที่มีความถูกต้องและเสถียรเมื่อมีการไหลแบบย้อนกลับ ($u < 0$) ผลที่ได้มีความถูกต้องเมื่อเทียบกับคำตอบแน่นอน จากการพัฒนาผลที่ได้แสดงให้เห็นว่า วิธีที่นำเสนอนี้สามารถนำไปประยุกต์ใช้ในการหาคำตอบในกรณีที่มีการไหลแบบแยกตัวซึ่งปกติการคำนวณด้วยวิธีตรง (direct method) ไม่สามารถทำได้ อย่างไรก็ตามเจตนาในการพัฒนานี้ไม่ต้องการที่จะไปแทนที่การคำนวณแบบย้อนกลับ (inverse method) แต่ต้องการแสดงให้เห็นถึงแนวทางที่สะดวกในการเทียบเคียง (couple) กับรูปแบบย้อนกลับทำให้การหาคำตอบได้รวดเร็วขึ้น

[1] *ผู้ช่วยศาสตราจารย์ ภาควิชาวิศวกรรมเครื่องกล*

Finite-Difference Scheme of the Incompressible Laminar Boundary-Layer Flow

Wanchai Asvapoositkul [1]

King Mongkut's University of Technology Thonburi, Bangmod, Toongkru, Bangkok 10140

Abstract

A time-accurate scheme has been developed for computing the incompressible laminar flow. The two-dimensional incompressible laminar boundary layer equations are marched in time by using a time stepping method. The momentum equation is solved by using central differencing in the direction normal to the body and upwind differencing in the other direction. The equation is solved by a tridiagonal matrix system. The main objective is to develop computational procedure which will provide accurate and stable solution when flow reversal ($u < 0$) occurs. The results compare favorable with an exact solution. It is deduced that the present method is quite accurate. The solutions showed the reverse flow velocity with the help of upwind scheme. The scheme is not intended to replace the inverse method and should be developed as an alternative to couple with the inverse mode.

[1] *Assistant Professor, Department of Mechanical Engineering.*

1. Introduction

In many application, we have to consider the flow in a boundary layer where there is either a favorable (falling) or adverse (rising) pressure gradient in the direction of the external stream. It is found experimentally that a falling pressure gradient has a stability effect on laminar flow in the boundary layer and tends to prolong the extent of the laminar region. Conversely, a rising pressure gradient has a destabilizing effect and tends to promote transition to turbulent flow in the boundary layer. Under extreme conditions with a severe adverse pressure gradient the velocity profile may reach the point where the velocity gradient, u_y, is zero at the wall. Under these conditions separation of the flow occurs.

In the standard method or "direct method" of solving the boundary layer equations, the boundary layer flow is computed by specifying the no-slip conditions at the wall and inviscid properties at the boundary layer edge. The parameters such as momentum thickness, displacement thickness, wall shear stress and skin friction are obtained as parts of the solutions. A finite-difference scheme which is efficiently solved for this type of flow may be seen in Asvapoositkul [1]. However, near and in reversed-flow regions, direct calculation procedures cannot solve the equations. This is because of the singularity of the standard boundary-layer formulation at separation. So the boundary-layer equations are solved in the inverse mode.

The prediction of flow separation is the most difficult aspects of the boundary layer approach. In recent years much effort has been devoted to develop methods for predicting separation of boundary layer. In the case when the reverse flow velocity is small compared with the outer velocity, one can remove the instability by using a simple technique known as the FLARE approximation. In this method the calculation procedure is the same as that in the direct method except that the streamwise convective term (e.g. $u\,u_x$) is neglected when the velocity is negative (i.e. $u<0$). Since the method is based on the assumption that the separation is small therefore the solutions are acceptable if the reverse flow is less than 10% of the local outer velocity [2].

Another method to solve the boundary layer separation when the FLARE approximation cannot be used is the so-called inverse approach. Catherall and Mangler [3] demonstrated that the boundary-layer equations are not singular at separation when displacement thickness is prescribed instead of free-stream velocity. This technique is known as the "inverse method". It is the boundary condition that differs between the direct and the inverse methods. In practice, we have to solve the equations by iteration until the specified displacement thickness (δ^*) is satisfied since u_e is not known before the boundary layer calculations are completed.

Once a solution is obtained by the inverse method, more accurate solutions of the boundary layer equations can be generated by using viscous-inviscid interaction methods. The best approach, at the moment, is the "interactive method" due to Veldman [4]. The essence of the method is that both the free-stream velocity and displacement thickness are treated as boundary condition that couple through the use of a Hilbert integral. The interactive boundary conditions that is combination of an external velocity and a displacement thickness describes how the outer potential flow reacts to the presence of the boundary layers.

An upwind scheme differs from the schemes described in Asvapoositkul [1] that this scheme is based on bi-directional marching technique. It's marching scheme changes according to the local values of velocity (e.g. u). This is because the stability constraints for boundary-layer flow depend upon the local values of velocities. This scheme is less straightforward to implement than simple marching schemes but it is attractive since it provides stable solutions when the crossflow is reversed (e.g. w<0). Many authors (e.g. Johnston [5], Steger and Van Dalsem [6], and Vatsa [7]) have successfully computed the boundary-layer using this scheme. Detail of the scheme will be discussed in discretization section 3 of this paper.

The purpose of this paper is to present a finite-difference scheme which is efficiently solved for reverse flow. The scheme is described in a time-accurate base on an upwind method.

2. Governing Equations and an Exact Solution

In the case of two-dimensional motion, the boundary-layer equation and their boundary conditions are given by [5]:

Continuity

$$u_x + v_y = 0 \tag{1}$$

x-momentum

$$u_t + u\,u_x + v\,u_y = u_e\,u_{e,x} + \nu\,u_{yy} \tag{2}$$

Boundary conditions

$$\text{at } y = 0 \;:\quad u = v = 0 \tag{3}$$

$$\text{at } y \to \infty \;:\quad u = u_e$$

When the velocity of the potential flow is proportional to a power of the length coordinate, x, measured from the stagnation point, i.e. for

$$u_e = u_\infty\,x^m \tag{4}$$

The transformation of the independent variable y, which leads to an ordinary differential equation is

$$\zeta = y\sqrt{\frac{m+1}{2}\frac{u_e}{\nu x}} \tag{5}$$

The velocity components are

$$u = u_e f' \tag{6}$$

$$v = -\sqrt{\frac{m+1}{2}\frac{u_e\nu}{x}}\left(f + \frac{m-1}{m+1}\zeta f'\right) \tag{7}$$

We obtain the following differential equation for $f(\zeta)$:

$$f''' + f f'' + \frac{2m}{m+1}(1 - f'^2) = 0 \tag{8}$$

Its boundary conditions are

$$\text{at } \zeta = 0 \qquad f = f' = 0 \tag{9}$$

$$\text{at } \zeta \to \infty \qquad f' = 1$$

Separation occurs for m = – 0.091 [8]. Equation (8) is an ordinary differential equation. The solution may be obtained by treating the third-order equation as a set of three simultaneous first-order equations, and the integration was done using the Runge-Kutta method with fourth-order accuracy. Detail of similar technique can be seen in Chow [9].

3. Finite Difference Scheme

To solve the boundary-layer equations using finite-difference scheme it is necessary to introduce coordinate transformations. For computational convenience, equation (1) and (2) are transformed from the physical x,y domain to a uniform $\xi(x)$, $\eta(x,y)$ computational domain:

Continuity equation

$$\xi_x u_\xi + \eta_x u_\eta + \eta_y v_\eta = 0 \tag{10}$$

x-momentum

$$u_t + u(\xi_x u_\xi + \eta_x u_\eta) + v(\eta_y u_\eta) = u_e \xi_x u_{e,\xi} + \nu\eta_y(\eta_{y\eta} u_\eta + \eta_y u_{\eta\eta}) \tag{11}$$

Boundary conditions

$$\text{at } \eta = 0 \quad : \qquad u = v = 0 \tag{12}$$

$$\text{at } \eta \to \infty \quad : \qquad u = u_e$$

Upwind Scheme

The upwind scheme is second-order accurate in space and first-order accurate in time. The scheme is applied in the momentum equation (11). The central difference operator (δ) is used in the η-direction and the forward difference operator (Δ) is used in time. For the ξ-direction, the backward difference operator (∇) is used when the value of u is positive and the forward difference operator (Δ) is used when the value of u is negative. Using a notation that space-time indices are not written unless changed (e.g., $u = u^{n}_{i,j}$, $u^{n+1} = u^{n+1}_{i,j}$, etc.), the first-order time-accurate method may be written as:

x-momentum

$$(\Delta_n u) + \frac{(\xi_x u + |\xi_x u|)}{2}(\nabla_\xi u) + \frac{(\xi_x u - |\xi_x u|)}{2}(\Delta_\xi u) + u\,\eta_x\,(\delta_\eta u^{n+1}) + v\,\eta_y\,(\delta_\eta u^{n+1})$$
$$= u_e\,\xi_x\,(\nabla_\xi u_e) + \nu\,\eta_y\,\eta_{y\eta}\,(\delta_\eta u^{n+1}) + \nu\,\eta_y\,\eta_y\,(\delta_{\eta\eta} u^{n+1}) \qquad \textbf{(13)}$$

The resulting difference equation can be written in the form

$$a\,u^{n+1}_{j-1} + b\,u^{n+1} + c\,u^{n+1}_{j+1} = d \qquad \textbf{(14)}$$

where $a = -\eta_x u - \eta_y v + \eta_y \eta_{y\eta} \nu - 2\,\eta_y \eta_y \dfrac{\nu}{\Delta\eta}$

$$b = 2\frac{\Delta\eta}{\Delta n} + 4\,\eta_y \eta_y \frac{\nu}{\Delta\eta}$$

$$c = \eta_x u + \eta_y v - \eta_y \eta_{y\eta} \nu - 2\,\eta_y \eta_y \frac{\nu}{\Delta\eta}$$

$$d = -\frac{\Delta\eta}{\Delta\xi}\frac{(\xi_x u + |\xi_x u|)}{2}(3u - 4u_{i-1} + u_{i-2}) - \frac{\Delta\eta}{\Delta\xi}\frac{(\xi_x u - |\xi_x u|)}{2}$$
$$(-3u + 4u_{i+1} - u_{i+2}) + 2\frac{\Delta\eta}{\Delta n}u + \frac{\Delta\eta}{\Delta\xi}u_e\,\xi_x\,(3u_e - 4u_{e,i-1} + u_{e,i-2})$$

This set of equations can be solved for the unknown u^{n+1} by a tridiagonal matrix. Once u^{n+1} is available, v^{n+1} is obtained from continuity equation in the discretised form.

$$v^{n+1} = v^{n+1}_{j-1} - 0.5\frac{\Delta\eta}{\Delta\xi}\frac{1}{\eta_y}(u^{n+1} - u^{n+1}_{i-1} + u^{n+1}_{j-1} - u^{n+1}_{i-1,j-1}) - \frac{\eta_x}{\eta_y}(u^{n+1} - u^{n+1}_{j-1}) \qquad \textbf{(15)}$$

When i = 1 equations (13) and (15) are calculated with first-order accurate in $\Delta\xi$. The solution at time n+1 at each downstream location can be achieved form the above equations. For steady flow solutions, the process is repeated until the difference between two successive values of u and v, for ξ-η plane, be less than a specified tolerance (i.e. $|u^{n+1} - u| \le \varepsilon$). Therefore, iteration is required. In practice it is more efficient to use a relaxation factor for updating the solutions.

4. Results and Discussions

The finite-difference scheme discussed in the previous section was used to obtain the solutions of the potential flow given by $u_e = u_\infty x^m$ for both positive and negative of m. If m is positive, the pressure gradient is negative or favorable, negative m denotes an unfavorable positive pressure gradient, and m = 0 denotes no pressure gradient. For laminar flow, the Reynolds number $Re = \frac{u_\infty L}{\nu} = 10^5$, where the characteristic length L = 1 m and u_∞ = 1 m/s. In this calculation the convergence criterion was taken as 1×10^{-4}. A 101x101 grid point of x and y-directions was used. The calculations were started without using Blasius solution but specifying the flow with no-slip at wall. The solutions from the present method are compared with those obtained by an exact solution [8]. It should be noted that the separation obtained from the present method occurs when m = -0.094. Fig. 1 shows a comparison of velocity profiles obtained by the method and those obtained by an exact solution.

As we can see from the Fig. 1, the calculated results in general agree quite well with an exact solution. For favorable pressure gradient, the value of m was set equal 0.5. For adverse pressure gradient, the value of m was set equal -0.90 since this is the lowest value that the solution can be obtained from the analytical method. Because p_ξ (i.e. $u_e u_{e,x}$) is positive, the velocity gradient u_η increase initially with increasing distance from the wall before reaching a maximum value at some intermediate value of η and subsequently falling off to zero at the outer edge of the boundary layer. Under this condition with a severe adverse pressure gradient the computations required more time steps than those in favorable pressure gradient to reach an acceptable level of convergence.

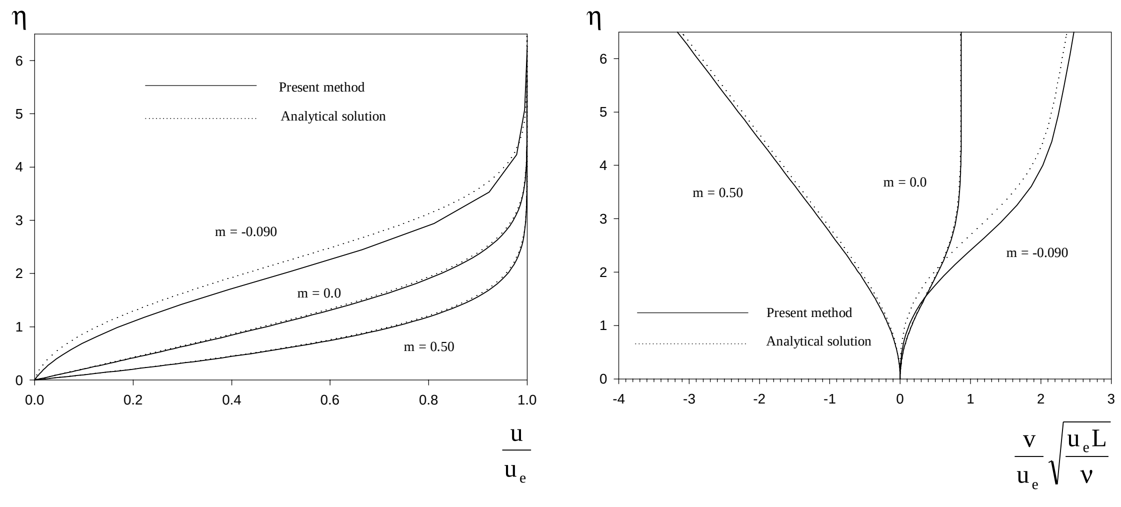

Fig. 1 Comparison of velocity profile for $u_e = u_\xi x^m$ from the present method and that from an exact solution.

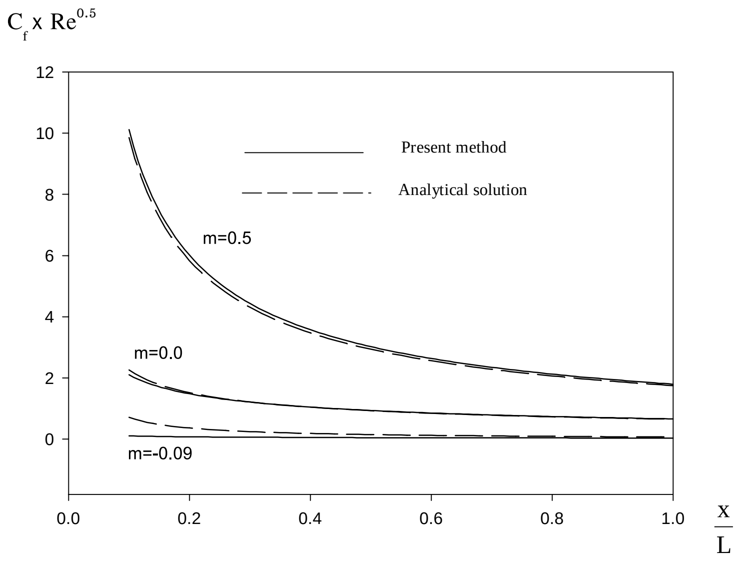

Fig. 2 Wall shearing stress

Fig. 2 shows the excellent agreement between the skin friction obtained from the present method and that from the analytical method with small exceptions near the leading edge of m = -0.90. The skin friction coefficient is calculated from

$$c_f = \frac{\nu\left[\frac{\partial u}{\partial y}\right]_{x,y=0}}{\frac{1}{2}u_{e,x}^2}$$

This is compared with the exact skin friction coefficient

$$c_f = 2f''\left[\frac{m+1}{2}\frac{\nu}{u_{e,x}x}\right]^{\frac{1}{2}}$$

In general, this agreement is within acceptable accuracy.

The results in Fig. 3 show that the numerical method has no difficulties in computing the region of reversed velocities. However the method shows poor convergence when the flow is reversed. Theoretically, the method is inaccurate beyond the flow separation point when the reverse flow is high. The present method is not intended to replace the inverse method but rather to demonstrate the robustness of the method to cope with a flow reversal.

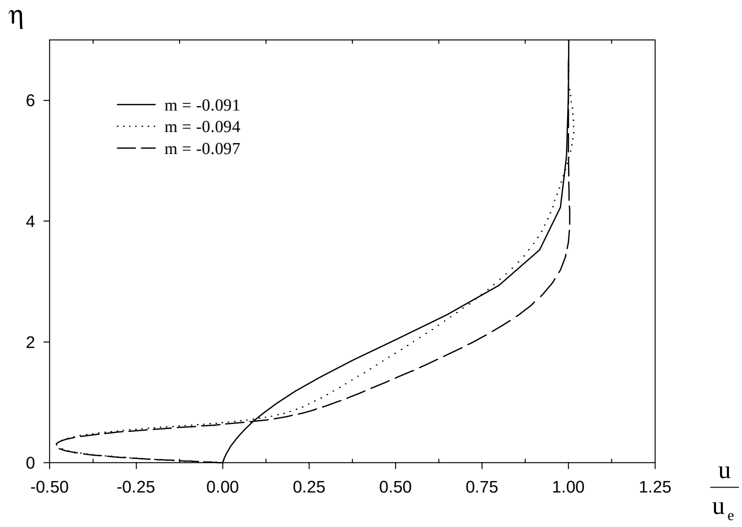

Fig. 3 Computed velocity profile for reversal flows.

5. Concluding Remarks

Concepts from a time-accurate for two-dimensional boundary-layer have been used to develop an upwind scheme for incompressible laminar flow. This scheme appears applicable with flow reversal is present. Computational results on a wide variety of flow situations on a flat-plate are validated and shown good agreement with analytical results. The scheme is robust and should be developed as an alternative to couple with the inverse method, and to verify, for complex flow such as three-dimensional/turbulent flow.

6. References

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Nomenclature

c_f	skin-friction coefficient
i, j	index of the grid point system
n	time step, index of time
t	time
u, v	velocity components
x, y	Cartesian co-ordinate
δ	central difference operator
$\Delta n, \Delta\xi, \Delta\eta$	grid spacing in time, ξ and η co-ordinates
Δ	forward difference operator
∇	backward difference operator
ε	convergence criteria
ζ	Blasius similarity variable

Subscript

e	outer edge of the boundary layer
i, j	denoting co-ordinate direction
x, y, ξ, η	derivative with respect to x, y, ξ, η
∞	upstream conditions

Superscript

$'$	differential equation