วิธีทางเลือกสำหรับคำนวณปริพันธ์ผสมในการแก้ปัญหา ศักย์สามมิติด้วยวิธีบาวน์ดะรีเอลิเมนต์

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บทคัดย่อ

ปริพันธ์ผสมซึ่งประกอบด้วยส่วนที่เป็นเอกฐานอย่างอ่อนและส่วนที่ไม่เป็นเอกฐานสามารถหาค่าได้ด้วยการแยกเป็นสองส่วนและหาค่าแต่ละส่วนด้วยวิธีต่างกัน บทความนี้นำเสนอเทคนิควิธีเชิงวิเคราะห์หรือกึ่งเชิงวิเคราะห์สำหรับคำนวณปริพันธ์เอกฐานอย่างอ่อนและใช้สูตรประมาณเกาส์เซียนสองมิติสำหรับคำนวณปริพันธ์ไม่เอกฐาน บทความนี้แสดงให้เห็นว่าปริพันธ์ผสมที่พบในการแก้ปัญหาศักย์สามมิติด้วยวิธีบาวน์ดะรีเอลิเมนต์สามารถแยกเป็นสองส่วนเพื่อใช้วิธีที่นำเสนอในบทความคำนวณได้ ประสิทธิผลของวิธีนี้ได้รับการสาธิตให้เห็นโดยการเปรียบเทียบผลเฉลยของปัญหาตัวอย่างที่ได้จากวิธีนี้และจากวิธีแปลงตัวแปร

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An Alternative Method for Calculating Mixed Surface Integrals in Solving the Three-dimensional Potential Problem by the Boundary Element Method

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Abstract

A mixed integral consisting of a weakly singular part and a non-singular part can be evaluated by separating it into the two parts and treating them differently. This paper presents the method that uses an analytical or semi-analytical technique for calculating the weakly singular integral, and the two-dimensional Gaussian quadrature for calculat ing the non-singular integral. It also shows how the separation of mixed integrals found in three-dimensional potential problems can be done. The effectiveness of the proposed method is demonstrated by comparing the solutions for a sample problem obtained from this method and from the variable transformation method.

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1. Introduction

A mixed integral consists of a weakly singular part and a non-singular part. An example of one-variable mixed integrals is

$$\int_0^1 \sqrt{1+x}\,\ln x\,\mathrm{d}x$$

It can be seen that the integrand becomes infinite at $x = 0$. Numerical integration of this integral using the one-dimensional Gaussian quadrature will yield an inaccurate result because the integrand is a singular function (i.e. its value is infinite within the limits of integral), which the Gaussian quadrature cannot handle well. Recommended method of evaluating this integral numerically [1] makes use of the one-dimensional logarithmic Gaussian quadrature. However, there is another way to calculate this integral. Taylor series expansion of $\sqrt{1+x}$ is

$$\sqrt{1+x} = 1+\frac{x}{2}+\ldots$$

or

$$\sqrt{1+x} = 1+P_1(x)$$

where $P_1(x)$ represents a polynomial that has zero coefficient for term of order 0. Therefore,

$$\int_0^1 \sqrt{1+x}\,\ln x\,\mathrm{d}x = \int_0^1 \ln x\,\mathrm{d}x + \int_0^1 \left(\sqrt{1+x}-1\right)\ln x\,\mathrm{d}x$$

Note that the first integral on the right hand side can be evaluated analytically:

$$\int_0^1 \ln x\,\mathrm{d}x = -1$$

The second integral is non-singular, and can be evaluated accurately by using the one-dimensional Gaussian quadrature. This method of calculating the mixed integrals may give better results than the method that uses the one-dimensional logarithmic Gaussian quadrature.

In this paper, two-variable mixed integrals are considered. A conventional method like the variable transformation method [1]-[4] deals with mixed integrals by using the Jacobian of transformation to cancel out the singularity before applying the two-dimensional Gaussian quadrature. However, an alternative method analogous to the method applied to the above one-variable mixed integral is presented here. Application of this method to a two-variable mixed integral yields a weakly singular integral, which can be evaluated analytically or semi-analytically, and the non-singular part, which can be evaluated numerically by using the two-dimensional Gaussian quadrature. The following sections describe analytical and semi-analytical evaluations of two-variable weakly singular integrals; separations of a two-variable mixed integral into a weakly singular part and a non-singular part; the variable transformation method of calculating two-variable mixed integrals; and the boundary element formulation of the three-dimensional heat conduction problem. Finally, the proposed method is compared with the variable transformation method in solving a sample problem by the boundary element method. It is shown that the proposed method gives a better result.

2. Weakly Singular Integrals

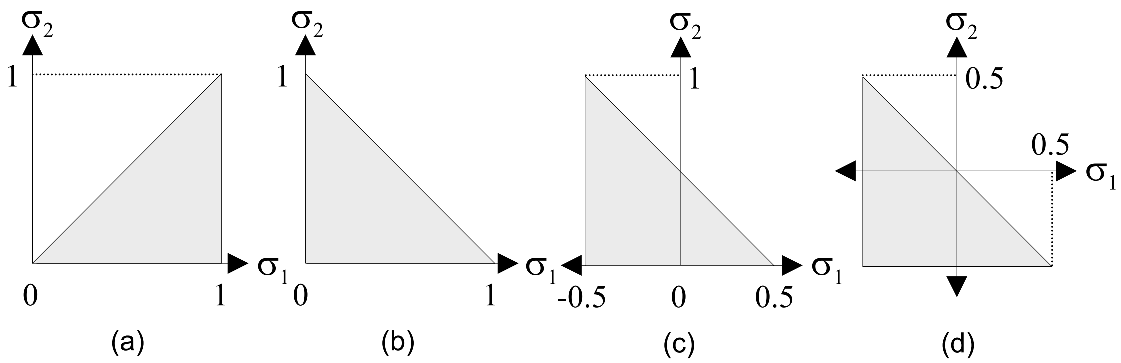

Fig. 1 Surface of integrals and coordinate systems for (a) Case 1, (b) Case 2, (c) Case 3, (d) Case 4

Weakly singular integrals $\iint_S f_i(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2$ (i = 1, 2) found in the three-dimensional potential problem have as their integrands the following functions:

$$f_1(\sigma_1, \sigma_2) = \frac{1}{\sqrt{A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2}} \tag{1}$$

$$f_2(\sigma_1, \sigma_2) = \frac{D\sigma_1^2 + E\sigma_1\sigma_2 + F\sigma_2^2}{\left(A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2\right)^{3/2}} \tag{2}$$

The origin of the (σ_1, σ_2) coordinate system is on the boundary of the surface S of the integrals. Although the integrands are infinite within the surface, these two integrals are finite. Their values can be obtained by a combination of the analytical and numerical techniques. The following formulas for indefinite integrals will be used for this purpose.

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \ln\left(b + 2ax + 2\sqrt{a^2x^2 + abx + ac}\right) \tag{3}$$

$$\int \frac{x}{\left(ax^2+bx+c\right)^{3/2}}\,\mathrm{d}x = \frac{-2(2c+bx)}{\left(4ac-b^2\right)\sqrt{ax^2+bx+c}} \tag{4}$$

$$\int \frac{x^2}{\left(ax^2+bx+c\right)^{3/2}}\,\mathrm{d}x = \frac{2bc-2\left(2ac-b^2\right)x}{a\left(4ac-b^2\right)\sqrt{ax^2+bx+c}} + \frac{1}{a^{3/2}}\ln\left(b+2ax+2\sqrt{a^2x^2+abx+ac}\right) \tag{5}$$

where $a \neq 0$. If the boundary element used is the six-node triangular element, there will be 4 cases of weakly singular integrals, corresponding to 4 different locations of the origin of the (σ_1, σ_2) coordinate system in the triangular surface S. Fig. 1 illustrates these cases. In each case, weakly singular integrals in Eqs. (1) and (2) can be evaluated analytically using the formulas in Eqs. (3)-(5).

Case 1

$$\int_0^1\int_0^{\sigma_1} \frac{1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2}}\,\mathrm{d}\sigma_2\mathrm{d}\sigma_1 =$$

$$\frac{1}{\sqrt{C}}\left\{\ln\left[\frac{B}{C}+2+2\sqrt{\frac{A+B+C}{C}}\right]-\ln\left[\frac{B}{C}+2\sqrt{\frac{A}{C}}\right]\right\} \tag{6}$$

$$\int_0^1\int_{\sigma_2}^{1} \frac{\sigma_1^2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}\,\mathrm{d}\sigma_1\mathrm{d}\sigma_2 = \frac{1}{A^{3/2}} - \frac{2B^2-4AC+2BC}{A\left(4AC-B^2\right)\sqrt{A+B+C}}$$
$$+\int_0^1 \frac{2B^2-4AC+2BC\sigma_2}{A\left(4AC-B^2\right)\sqrt{A+B\sigma_2+C\sigma_2^2}}\,\mathrm{d}\sigma_2$$
$$+\frac{1}{A^{3/2}}\int_0^1 \ln\left[\frac{B\sigma_2+2A+2\sqrt{A^2+AB\sigma_2+AC\sigma_2^2}}{B+2A+2\sqrt{A^2+AB+AC}}\right]\mathrm{d}\sigma_2 \tag{7}$$

$$\int_0^1\int_0^{\sigma_1} \frac{\sigma_1\sigma_2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}\,\mathrm{d}\sigma_2\mathrm{d}\sigma_1 = \frac{4\sqrt{A}}{\left(4AC-B^2\right)} - \frac{2(2A+B)}{\left(4AC-B^2\right)\sqrt{A+B+C}} \tag{8}$$

$$\int_0^1\int_0^{\sigma_1} \frac{\sigma_2^2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}\,\mathrm{d}\sigma_2\mathrm{d}\sigma_1 = \frac{1}{C^{3/2}}\ln\left[\frac{B+2C+2\sqrt{C^2+BC+AC}}{B+2\sqrt{AC}}\right]$$
$$-\frac{2B\sqrt{A}}{C\left(4AC-B^2\right)} + \frac{2B^2-4AC+2AB}{C\left(4AC-B^2\right)\sqrt{A+B+C}} \tag{9}$$

Case 2

$$\int_0^1\int_0^{1-\sigma_1}\frac{1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2}}\,d\sigma_2 d\sigma_1 = -\frac{1}{\sqrt{C}}\left\{\ln\left[\frac{B}{C}+2\sqrt{\frac{A}{C}}\right]-1\right\}$$
$$+\frac{1}{\sqrt{C}}\int_0^1\ln\left[\left(\frac{B}{C}-2\right)\sigma_1+2+2\sqrt{\left(\frac{A+B+C}{C}\right)\sigma_1^2+\left(\frac{B-2C}{C}\right)\sigma_1+1}\right]d\sigma_1 \quad (10)$$

$$\int_0^1\int_0^{1-\sigma_2}\frac{\sigma_1^2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}\,d\sigma_1 d\sigma_2 = \frac{1}{A^{3/2}}\left\{1-\ln\left[B+2\sqrt{AC}\right]\right\}$$
$$+\frac{1}{A^{3/2}}\int_0^1\ln\left[B\sigma_2-2A\sigma_2+2A+2\sqrt{AC\sigma_2^2+AB(1-\sigma_2)\sigma_2+A\left(1-\sigma_2^2\right)}\right]d\sigma_2 \quad (11)$$

$$\int_0^1\int_0^{1-\sigma_1}\frac{\sigma_1\sigma_2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}\,d\sigma_2 d\sigma_1 = \frac{4\sqrt{A}}{\left(4AC-B^2\right)}-$$
$$\int_0^1\frac{2\left(2A\sigma_1-B\sigma_1+B\right)}{\left(4AC-B^2\right)\sqrt{A\sigma_1^2+B\sigma_1(1-\sigma_1)+C\left(1-\sigma_1^2\right)}}\,d\sigma_1 \quad (12)$$

$$\int_0^1\int_0^{1-\sigma_1}\frac{\sigma_2^2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}\,d\sigma_2 d\sigma_1 = \frac{1}{C^{3/2}}\left\{1-\ln\left[B+2\sqrt{AC}\right]\right\}$$
$$+\frac{1}{C^{3/2}}\int_0^1\ln\left[B\sigma_1-2C\sigma_1+2C+2\sqrt{AC\sigma_1^2+BC(1-\sigma_1)\sigma_1+C\left(1-\sigma_1^2\right)}\right]d\sigma_1 \quad (13)$$

Case 3

$$\int_{-0.5}^{0.5}\int_0^{0.5-\sigma_1}\frac{1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2}}\,d\sigma_2 d\sigma_1 = -\frac{1}{\sqrt{C}}\left\{\frac{1}{2}\ln\left[\frac{4AC-B^2}{C^2}\right]-1-\ln 2\right\}$$
$$+\frac{1}{\sqrt{C}}\int_{-0.5}^{0.5}\ln\left[\left(\frac{B}{C}-2\right)\sigma_1+1+2\sqrt{\left(\frac{A-B+C}{C}\right)\sigma_1^2+\left(\frac{B-2C}{2C}\right)\sigma_1+\frac{1}{4}}\right]d\sigma_1 \quad (14)$$

$$\int_{-0.5}^{0.5}\int_0^{1-\sigma_2}\frac{\sigma_1^2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}\,d\sigma_1 d\sigma_2 =$$
$$\int_0^1\frac{\left(4AC+2BC-2B^2\right)\sigma_2+B^2-2AC}{A\left(4AC-B^2\right)\sqrt{C\sigma_2^2+B\sigma_2(0.5-\sigma_2)+A(0.5-\sigma_2)^2}}\,d\sigma_2$$
$$-\int_0^1\frac{2BC\sigma_2-B^2+2AC}{A\left(4AC-B^2\right)\sqrt{C\sigma_2^2-0.5B\sigma_2+0.25A}}\,d\sigma_2$$
$$+\frac{1}{A^{3/2}}\int_0^1\ln\left[\frac{B\sigma_2-2A\sigma_2+A+2\sqrt{AC\sigma_2^2+AB\sigma_2(1-\sigma_2)+A^2(1-\sigma_2)^2}}{B\sigma_2-A+2\sqrt{AC\sigma_2^2-0.5AB\sigma_2+0.25A^2}}\right]d\sigma_2 \quad (15)$$

$$\int_{-0.5}^{0.5}\int_{0}^{0.5-\sigma_1}\frac{\sigma_1\sigma_2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}d\sigma_2 d\sigma_1 = \frac{4\sqrt{A}}{4AC-B^2} -$$

$$\int_{-0.5}^{0.5}\frac{4A\sigma_1-2B\sigma_1+B}{\left(4AC-B^2\right)\sqrt{A\sigma_1^2+B\sigma_1(1-\sigma_1)+C(1-\sigma_1)^2}}d\sigma_1 \qquad (16)$$

$$\int_{-0.5}^{0.5}\int_{0}^{1-\sigma_1}\frac{\sigma_2^2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}d\sigma_2 d\sigma_1 = \frac{1}{C^{3/2}}\left\{-\ln\sqrt{4AC-B^2}+\ln 2+1\right\}$$

$$-\frac{2B\sqrt{A}}{C\left(4AC-B^2\right)}$$

$$+\frac{1}{C^{3/2}}\int_{-0.5}^{0.5}\ln\left[B\sigma_1-2C\sigma_1+C+2\sqrt{AC\sigma_1^2+BC\sigma_1(0.5-\sigma_1)+C(0.5-\sigma_1)^2}\right]d\sigma_1 \qquad (17)$$

Case 4

$$\int_{-0.5}^{0.5}\int_{-0.5}^{-\sigma_1}\frac{1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2}}d\sigma_2 d\sigma_1 = \frac{1}{\sqrt{C}}\left\{\frac{1}{2}\ln\left[\frac{4AC-B^2}{C^2}\right]+1+\ln 2\right\}$$

$$-\frac{1}{\sqrt{C}}\int_{-0.5}^{0.5}\left\{\ln\left[\frac{B\sigma_1}{C}-1+2\sqrt{\frac{4A\sigma_1^2-2B\sigma_1+C}{4C}}\right]-2\ln 2\right\}d\sigma_1 \qquad (18)$$

$$\int_{-0.5}^{0.5}\int_{-0.5}^{-\sigma_1}\frac{\sigma_1^2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}d\sigma_1 d\sigma_2 = \frac{1}{A^{3/2}}\left\{\ln\sqrt{4AC-B^2}-\ln 2-1\right\}+$$

$$\frac{2BC+4AC-2B^2}{A\left(4AC-B^2\right)\sqrt{A-B+C}}-\int_{-0.5}^{0.5}\frac{2BC\sigma_2+2AC-B^2}{A\left(4AC-B^2\right)\sqrt{C\sigma_2^2-0.5B\sigma_2+0.25A}}d\sigma_2 \qquad (19)$$

$$\int_{-0.5}^{0.5}\int_{-0.5}^{-\sigma_1}\frac{\sigma_1\sigma_2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}d\sigma_2 d\sigma_1 = \frac{-2(2A-B)}{\left(4AC-B^2\right)\sqrt{A-B+C}}$$

$$+\int_{-0.5}^{0.5}\frac{4A\sigma_1-B}{\left(4AC-B^2\right)\sqrt{A\sigma_1^2-0.5B\sigma_1+0.25C}}d\sigma_1 \qquad (20)$$

$$\int_{-0.5}^{0.5}\int_{-0.5}^{-\sigma_1}\frac{\sigma_2^2}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2\right)^{3/2}}d\sigma_2 d\sigma_1 = \frac{1}{C^{3/2}}\left\{\ln\sqrt{4AC-B^2}-\ln 2-1\right\}+$$

$$\frac{2AB+4AC-2B^2}{C\left(4AC-B^2\right)\sqrt{A-B+C}}-\int_{-0.5}^{0.5}\frac{2AB\sigma_1+2AC-B^2}{C\left(4AC-B^2\right)\sqrt{A\sigma_1^2-0.5B\sigma_1+0.25C}}d\sigma_1 \qquad (21)$$

Note that one-variable integrals in Eqs. (7), (10)-(21) are non-singular integrals; their integrands remain finite within the limits of the integrals. Hence, they can be evaluated numerically by using the one-dimensional Gaussian quadrature.

$$\int_{-1}^{1} f(x)\mathrm{d}x = \sum_{i=1}^{n_g} w_i f(x_i) \tag{22}$$

where x_i is a Gauss point. w_i is the weight, and n_g is the number of Gauss points.

3. Mixed Integrals

Mixed integrals in the boundary element formulation of the three-dimensional potential problem have the following forms:

$$\iint_S g_1(\sigma_1,\sigma_2)\mathrm{d}\sigma_1\mathrm{d}\sigma_2 = \iint_S \frac{1+P_1(\sigma_1,\sigma_2)}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2+P_3(\sigma_1,\sigma_2)}}\mathrm{d}\sigma_1\mathrm{d}\sigma_2 \tag{23}$$

$$\iint_S g_2(\sigma_1,\sigma_2)\mathrm{d}\sigma_1\mathrm{d}\sigma_2 = \iint_S \frac{D\sigma_1^2+E\sigma_1\sigma_2+F\sigma_2^2+P_3(\sigma_1,\sigma_2)}{\left(A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2+P_3(\sigma_1,\sigma_2)\right)^{3/2}}\mathrm{d}\sigma_1\mathrm{d}\sigma_2 \tag{24}$$

where $P_n(\sigma_1, \sigma_2)$ represents a polynomial that has zero coefficients for terms of order $n-1$ or lower. The integrand in Eq. (23) can be separated into two terms as follows

$$\frac{1+P_1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2+P_3}} = \frac{1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2+P_3}} + \frac{P_1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2+P_3}} \tag{25}$$

The second term on the right hand side of Eq. (25) does not become infinity as (σ_1, σ_2) equals (0, 0) because the polynomials in the numerator and the denominator are of the same order. Therefore, the integral of this term is non-singular. The first term on the right hand side of Eq. (25) can be rewritten as

$$\begin{aligned}\frac{1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2+P_3}} &= \frac{1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2}}\sqrt{\frac{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2}{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2+P_3}} \\ &= \frac{1}{\sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2}}\sqrt{1-\frac{P_3}{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2+P_3}}\end{aligned} \tag{26}$$

But $P_3(\sigma_1,\sigma_2) << A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2$ in the limit $(\sigma_1, \sigma_2) \to (0, 0)$. Therefore,

$$\lim_{(\sigma_1,\sigma_2)\to(0,0)} \sqrt{1 - \frac{P_3}{A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2 + P_3}} = 1 - \frac{P_3}{2\left(A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2 + P_3\right)} \quad (27)$$

Combining Eqs. (26) and (27) results in

$$\frac{1}{\sqrt{A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2 + P_3}} = \frac{1}{\sqrt{A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2}} - \frac{1}{2\sqrt{A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2}}\left(\frac{P_3}{A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2 + P_3}\right) \quad (28)$$

The second term on the right hand side of Eq. (28) is finite as (σ_1, σ_2) approaches (0, 0) because the polynomials in the numerator and the denominator are of the same order. Consequently, the integral having this term as its integrand is non-singular. This leaves the integral of the first term as the only weakly singular integral. The original integral can now be written as the sum of the weakly singular part and the non-singular part.

$$\iint_S g_1(\sigma_1,\sigma_2)\mathrm{d}\sigma_1\mathrm{d}\sigma_2 = \iint_S \frac{1}{\sqrt{A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2}}\mathrm{d}\sigma_1\mathrm{d}\sigma_2 + \iint_S \left(g_1(\sigma_1,\sigma_2) - \frac{1}{\sqrt{A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2}}\right)\mathrm{d}\sigma_1\mathrm{d}\sigma_2 \quad (29)$$

Similarly, it can be shown that

$$\iint_S g_2(\sigma_1,\sigma_2)\mathrm{d}\sigma_1\mathrm{d}\sigma_2 = \iint_S \frac{D\sigma_1^2 + E\sigma_1\sigma_2 + F\sigma_2^2}{\left(A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2\right)^{3/2}}\mathrm{d}\sigma_1\mathrm{d}\sigma_2 + \iint_S \left(g_2(\sigma_1,\sigma_2) - \frac{D\sigma_1^2 + E\sigma_1\sigma_2 + F\sigma_2^2}{\left(A\sigma_1^2 + B\sigma_1\sigma_2 + C\sigma_2^2\right)^{3/2}}\right)\mathrm{d}\sigma_1\mathrm{d}\sigma_2 \quad (30)$$

The first integrals on the right hand sides of Eqs. (29) and (30) can be evaluated by the technique shown in the previous section, whereas the second integrals can be computed by using the following transformation

$$\sigma_1 = \frac{1}{4}(1+\rho_1)(1-\rho_2) \quad (31)$$

$$\sigma_2 = \frac{1}{2}(1+\rho_2) \quad (32)$$

As a result, the surface of the integral is changed from right triangle to square.

$$\iint_S g(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 = \int_{-1}^{1}\int_{-1}^{1} g(\rho_1, \rho_2) \frac{1}{8}(1 - \rho_2) d\rho_1 d\rho_2 \quad (33)$$

and the two-dimensional Gaussian quadrature can be applied.

$$\int_{-1}^{1}\int_{-1}^{1} f(x, y) dx dy = \sum_{i=1}^{n_g}\sum_{j=1}^{n_g} w_i w_j f(x_i, y_j) \quad (34)$$

4. Variable Transformation Method of Evaluating Mixed Integrals

The mixed integrals in Eqs. (23) and (24) can be calculated [1] by dividing the domain of the integrals into triangles of which vertices are the locations of singularity (i.e. the integrand becomes infinite). Fig. 2 shows how a six-node triangular element is divided when the point of singularity is at each of the 6 nodes.

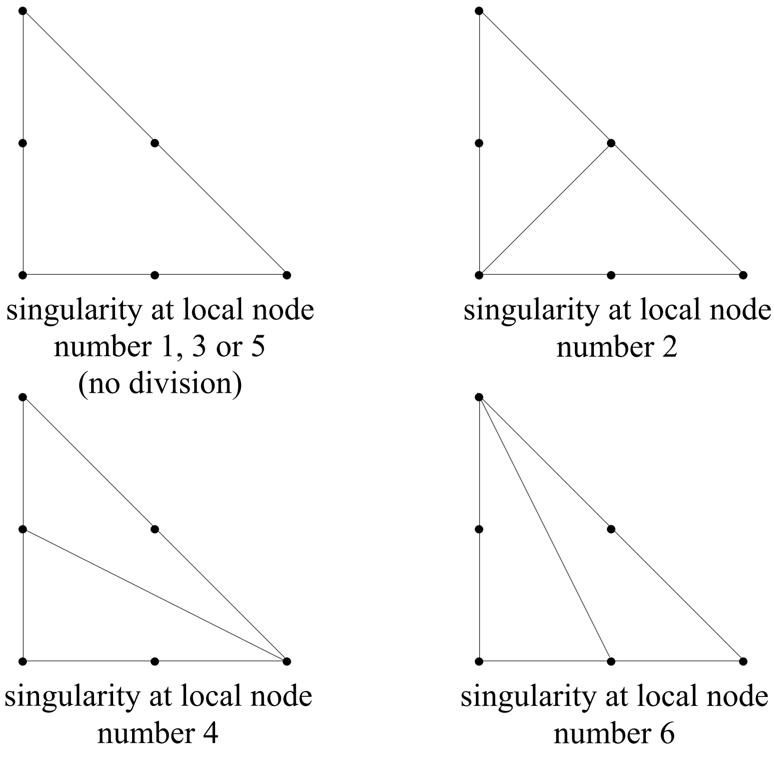

Fig. 2 Division of six-node triangular element in a conventional method of evaluating mixed integrals when the point of singularity is at each of the 6 local nodes

Mixed integrals over a triangle of which one of its vertices is the location of singularity can be done by transforming the triangle into a square as shown in Fig. 3. The transformation formulas are

$$\sigma_1 = \frac{1}{4}(1+\sigma_1')[b_1 + a_1 + \sigma_2'(b_1 - a_1)] \tag{35}$$

$$\sigma_2 = \frac{1}{4}(1+\sigma_1')[b_2 + a_2 + \sigma_2'(b_2 - a_2)] \tag{36}$$

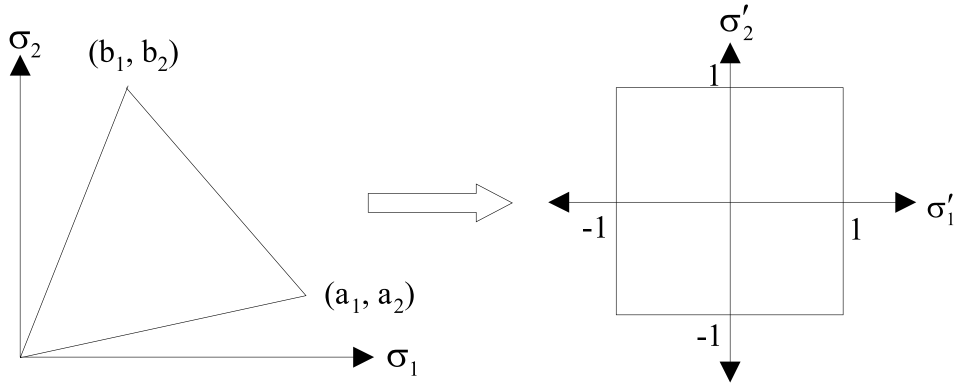

Fig. 3 Transformation of triangle in (σ_1, σ_2) coordinates into square in (σ_1', σ_2') coordinates

After transformation, integrals in Eqs. (23) and (24) becomes

$$\iint_S g_1(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 = \int_{-1}^{1}\int_{-1}^{1} g_1(\sigma_1',\sigma_2')J'(\sigma_1',\sigma_2')\mathrm{d}\sigma_1'\mathrm{d}\sigma_2' \tag{37}$$

$$\iint_S g_2(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 = \int_{-1}^{1}\int_{-1}^{1} g_2(\sigma_1',\sigma_2')J'(\sigma_1',\sigma_2')\mathrm{d}\sigma_1'\mathrm{d}\sigma_2' \tag{38}$$

Since $J' \rightarrow 0$ as g_1 and $g_2 \rightarrow \infty$, integrals on the right hand sides of Eqs. (37) and (38) are non-singular and can be calculated using the two-dimensional Gaussian quadrature (Eq. (34)).

5. Three-dimensional Potential Problem

For clarity, the potential problem to be considered is the steady-state heat conduction problem. The boundary integral equation for the three-dimensional heat conduction problem is [1]

$$a_i T(\vec{r}_i) = \int_S \frac{\dot{q}(\vec{r})}{k} G(\vec{r} - \vec{r}_i)\mathrm{d}A - \int_S T(\vec{r})\, \vec{n}\vec{\nabla} G(\vec{r} - \vec{r}_i)\mathrm{d}A \tag{39}$$

where i is the global node number, T is temperature, $\dot{q}$ is heat flux entering the domain in the direction normal to the surface of the domain, k is thermal conductivity, $\vec{n}$ is the unit normal vector pointing outward from the surface of the domain, a_i is coefficient that depends on position vector $\vec{r}_i$, and the fundamental solution G is

$$G(\vec{r}-\vec{r}_i) = \frac{1}{4k\pi|\vec{r}-\vec{r}_i|} \tag{40}$$

Suppose surface S is divided into M boundary elements. Multiplying Eq. (39) by k gives

$$ka_iT(\vec{r}_i) = \sum_{k=1}^{M}\int_{S_k}\dot{q}(\vec{r})G(\vec{r}-\vec{r}_i)\mathrm{d}A - \sum_{k=1}^{M}\int_{S_k}T(\vec{r})\,k\vec{n}\vec{\nabla}G(\vec{r}-\vec{r}_i)\mathrm{d}A \tag{41}$$

Next transform variables of integrals to natural coordinates (s_1, s_2), making use of the Jacobian of transformation $J\ (s_1, s_2)$.

$$ka_iT(\vec{r}_i) = \sum_{k=1}^{M}\iint_{S_k}\dot{q}(\vec{r})G(\vec{r}-\vec{r}_i)J(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2 -$$

$$\sum_{k=1}^{M}\iint_{S_k}T(\vec{r})\,k\vec{n}\vec{\nabla}G(\vec{r}-\vec{r}_i)J(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2 \tag{42}$$

T and $\dot{q}$ can be related to temperature and heat flux components at boundary nodes by interpolating functions.

$$\dot{q}(s_1,s_2) = \sum_{l=1}^{m}N_l(s_1,s_2)\dot{q}_{k,l} \tag{43}$$

$$T(s_1,s_2) = \sum_{l=1}^{m}N_l(s_1,s_2)T_{k,l} \tag{44}$$

where l is the local node number, and m is the number of nodes in an element. Inserting Eqs. (43) and (44) into Eq. (42) results in

$$ka_iT(\vec{r}_i) = \sum_{k=1}^{M}\sum_{l=1}^{m}\dot{q}_{k,l}\iint_{S_k}G(\vec{r}-\vec{r}_i)J(s_1,s_2)N_l(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2 -$$

$$\sum_{k=1}^{M}\sum_{l=1}^{m}T_{k,l}\iint_{S_k}k\vec{n}\vec{\nabla}G(\vec{r}-\vec{r}_i)J(s_1,s_2)N_l(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2 \tag{45}$$

There are two types of mixed integrals in Eq. (45). The first type occurs when the global node number i is at the same location as the local node number l in element k. The corresponding mixed integral is

$$I_1 = \iint_{S_k}\frac{J(s_1,s_2)N_l(s_1,s_2)}{4\pi k|\vec{r}-\vec{r}_i|}\mathrm{d}s_1\mathrm{d}s_2 \tag{46}$$

The second type occurs when the global node number *i* is in element *k*, but is not at the same location as the local node number *l* in element *k*. The corresponding mixed integral is

$$I_2 = \iint_{S_k} \frac{-1}{4\pi|\vec{r}-\vec{r}_i|^3}\left[(x-x_i)J_x+(y-y_i)J_y+(z-z_i)J_z\right]N_l(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2 \quad (47)$$

where $J_x = \frac{\partial y}{\partial s_1}\frac{\partial z}{\partial s_2}-\frac{\partial y}{\partial s_2}\frac{\partial z}{\partial s_1}$ (48)

$$J_y = \frac{\partial z}{\partial s_1}\frac{\partial x}{\partial s_2}-\frac{\partial z}{\partial s_2}\frac{\partial x}{\partial s_1} \quad (49)$$

$$J_z = \frac{\partial x}{\partial s_1}\frac{\partial y}{\partial s_2}-\frac{\partial x}{\partial s_2}\frac{\partial y}{\partial s_1} \quad (50)$$

$$J = \sqrt{J_x^2+J_y^2+J_z^2} \quad (51)$$

In order for I_1 and I_2 to be in the form of Eqs. (23) and (24), the (s_1, s_2) coordinate system must be transformed to an appropriate coordinate system (σ_1, σ_2), which depends on the type of boundary element and the local node number. As a result of such transformation,

$$N_l = \begin{cases} 1+P_1(\sigma_1,\sigma_2) & \text{if global node } i \text{ is at the same location as local node } l \text{ in element } k \\ c\sigma_1+d\sigma_2+P_2(\sigma_1,\sigma_2) & \text{if global node } i \text{ is at different location from local node } l \text{ in element } k \end{cases} \quad (52)$$

$$x-x_i = e_{x1}\sigma_1+e_{x2}\sigma_2+P_2(\sigma_1,\sigma_2) \quad (53)$$

$$y-y_i = e_{y1}\sigma_1+e_{y2}\sigma_2+P_2(\sigma_1,\sigma_2) \quad (54)$$

$$z-z_i = e_{z1}\sigma_1+e_{z2}\sigma_2+P_2(\sigma_1,\sigma_2) \quad (55)$$

$$|\vec{r}-\vec{r}_i| = \sqrt{A\sigma_1^2+B\sigma_1\sigma_2+C\sigma_2^2+P_3(\sigma_1,\sigma_2)} \quad (56)$$

$$A = e_{x1}^2+e_{y1}^2+e_{z1}^2 \quad (57)$$

$$B = 2(e_{x1}e_{x2}+e_{y1}e_{y2}+e_{z1}e_{z2}) \quad (58)$$

$$C = e_{x2}^2+e_{y2}^2+e_{z2}^2 \quad (59)$$

$$J_x = e'_{y1}e'_{z2}-e'_{y2}e'_{z1}+P_1(\sigma_1,\sigma_2) \quad (60)$$

$$J_y = e'_{y1}e'_{z2}-e'_{y2}e'_{z1}+P_1(\sigma_1,\sigma_2) \quad (61)$$

$$J_z = e'_{x1}e'_{y2}-e'_{x2}e'_{y1}+P_1(\sigma_1,\sigma_2) \quad (62)$$

Coefficients $e_{\xi 1}$, $e_{\xi 2}$, $e'_{\xi 1}$ and $e'_{\xi 2}$ $(\xi = x, y, z)$ depend on the type of boundary element and the local node number. For the six-node triangular element, their values are shown in Table 1.

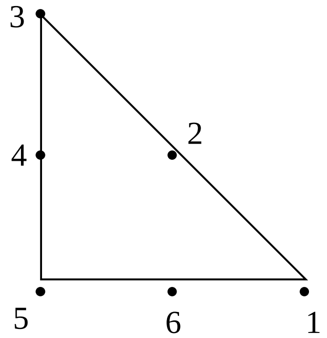

Fig. 4 Local node numbers in six-node triangular element

Table 1 Expressions of σ_1, σ_2, $e_{\xi 1}$, $e_{\xi 2}$, $e'_{\xi 1}$ and $e'_{\xi 2}$ for the 6-node triangular element

l	σ_1	σ_2	$e_{\xi 1}$	$e_{\xi 2}$	$e'_{\xi 1}$	$e'_{\xi 2}$
1	$1 - s_1$	s_2	$-3\xi_1 - \xi_5 + 4\xi_6$	$4\xi_2 - \xi_3 + \xi_5 - 4\xi_6$	$-e_{\xi 1}$	$e_{\xi 2}$
2	$s_1 - 0.5$	$s_2 - 0.5$	$\xi_1 + 2\xi_2 - 2\xi_4 + \xi_5 - 2\xi_6$	$2\xi_2 + \xi_3 - 2\xi_4 + \xi_5 - 2\xi_6$	$e_{\xi 1}$	$e_{\xi 2}$
3	$1 - s_2$	s_1	$-3\xi_3 + 4\xi_4 - \xi_5$	$-\xi_1 + 4\xi_2 - 4\xi_4 + \xi_5$	$e_{\xi 2}$	$-e_{\xi 1}$
4	$s_2 - 0.5$	s_1	$\xi_3 - \xi_5$	$-\xi_1 + 2\xi_2 - 2\xi_4 - \xi_5 + 2\xi_6$	$e_{\xi 2}$	$e_{\xi 1}$
5	s_1	s_2	$-\xi_1 - 3\xi_5 + 4\xi_6$	$-\xi_1 + 4\xi_4 - 3\xi_5$	$e_{\xi 1}$	$e_{\xi 2}$
6	$s_1 - 0.5$	s_2	$\xi_1 - \xi_5$	$2\xi_2 - \xi_3 + 2\xi_4 - \xi_5 - 2\xi_6$	$e_{\xi 1}$	$e_{\xi 2}$

ξ_1, ξ_2, …, ξ_6 are coordinates of local node numbers 1, 2, …, 6, respectively. The arrangement of 6 local node numbers in a six-node triangular element is shown in Fig. 4. Coefficients c and d in Eq. (52) also depend on the type of boundary element and the local node number. For the six-node triangular element, their values are given in Table 2

Table 2 Expressions of *c* and *d* for the 6-node triangular element

		Global node i is at the same location as local node l					
		$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$
N_1	c		1	0	0	-1	1
	d		0	-1	-1	0	0
N_2	c	0		0	0	0	0
	d	4		4	2	0	2
N_3	c	0	0		1	0	0
	d	-1	1		0	-1	-1
N_4	c	0	-2	4		0	0
	d	0	-2	-4		4	2
N_5	c	-1	1	-1	-1		-1
	d	1	1	1	-1		-1
N_6	c	4	-2	0	0	4	
	d	-4	-2	0	2	0	

6. Comparison between the Variable Transformation Method and the Alternative Method

The alternative method of evaluating mixed integrals is to be compared with the variable transformation method described previously. The boundary element method is used to solve the heat conduction problem illustrated in Fig. 5. The hollow cylinder has the inner radius of 4, the outer radius of 12 and the length of 3. The temperature of the inner surface is $100\ln(4)$, and the temperature of the outer surface is $100\ln(12)$, whereas the two ends are perfectly insulated. The material of the cylinder has the thermal conductivity of 1. Due to the symmetry of the problem, only a quarter of the cylinder (section with line texture) is considered. The six surfaces of the section are divided into a total of 76 six-node triangular elements. The distribution of boundary elements among the six surfaces is indicated in Fig. 5.

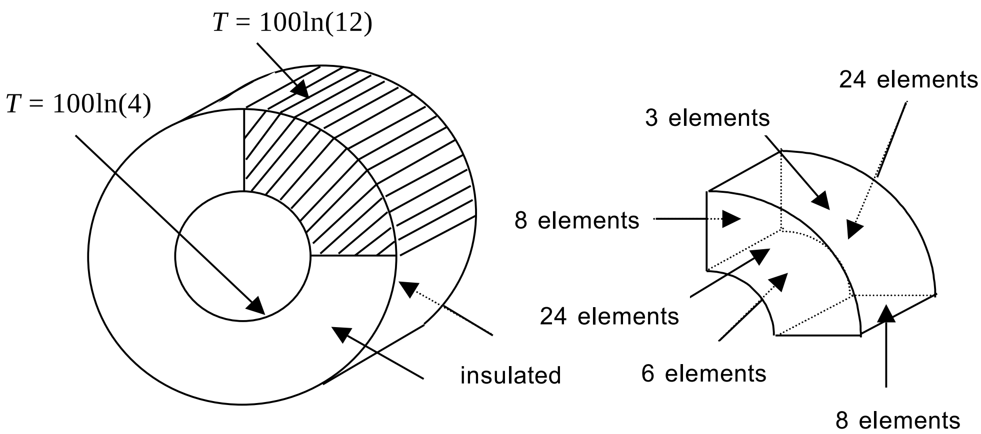

Fig. 5 Sample heat conduction problem

The exact solution for this problem is

$$T(r) = 100\ln(r) \tag{63}$$

where r is the radial distance from the centerline of the hollow cylinder. This means that the uniform heat flux of 25 leaves the inner surface, and the uniform heat flux of 8.3333 enters the outer surface. In addition to the exact solution, numerical solution to the problem can be obtained by using the boundary element method. If the boundary condition for the problem includes temperatures on the inner surface and the outer surface of the problem domain as shown in Fig. 5, and zero heat flux on other four surfaces, heat flux on the inner surface and the outer surface can be calculated. The accuracy of the numerical solution is estimated by computing the following error.

$$\varepsilon_i = \left| \frac{\dot{q}_i - \dot{q}_{\text{exact},i}}{\dot{q}_{\text{exact},i}} \right| \times 100 \tag{64}$$

where $\dot{q}_i$ is heat flux component at node i on the inner surface or the outer surface, and $\dot{q}_{\text{exact},i}$ is the exact heat flux component at the same node. Average error ε is given by

$$\varepsilon = \frac{1}{N_{io}} \sum_{i=1}^{N_{io}} \varepsilon_i \tag{65}$$

where N_{io} is the number of nodes on the inner surface and the outer surface.

Average errors are calculated for numerical solutions obtained from the boundary element method with the evaluation of weakly singular integrals performed by the variable transformation method and the alternative method. Table 3 shows that increasing n_g results in less average errors for both methods. Recall that n_g is the number of Gauss points in numerical integration of one-variable integrals (Eq. (22)), and n_g^2 is the number of Gauss points in numerical integration of two-variable integrals (Eq. (34)). When average errors are compared at the same n_g, it can be seen that the proposed method yields more accurate solutions.

Table 3 Comparison between the variable transformation method and the alternative method

n_g	Average error (%)	
	Variable Transformation Method	**Alternative Method**
2	21.3211	5.584562
4	4.058598	0.326068
6	1.593568	0.069178
8	0.918393	0.063236
10	0.635773	0.061101

7. Conclusions

The alternative method calculates a two-variable mixed integral by separating it into a weakly singular integral, which can be treated analytically or semi-analytically, and a non-singular integral, which can be treated numerically with the two-dimensional Gaussian quadrature. Mixed integrals under consideration are found in the boundary element formulation of the three-dimensional potential problem. The variable transformation method for calculating these integrals eliminates the singularity of integrands and changes the surface of integrals to a square, for which the two-dimensional Gaussian quadrature can be applied. The alternative method and the variable transformation method are used to evaluate mixed integrals in solving a three-dimensional heat conduction problem with the boundary element method. It is found that the alternative method yields more accurate results.

8. Acknowledgement

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9. References

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