การทำแบบจำลองไฟไนต์เอลิเมนต์สำหรับวัสดุภายใต้การอัดขึ้นรูปทรงกระบอกแบบไปข้างหน้าสมมาตรรอบแกน; ส่วนที่ 1: รูปแบบสมการการไหล

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บทคัดย่อ

บทความวิจัยฉบับนี้เกี่ยวข้องกับการประยุกต์ใช้เทคนิคทางไฟไนต์เอลิเมนต์เพื่อทำแบบจำลองกระบวนการแปรรูปของวัสดุแบบก้อน ประกอบด้วยสองส่วนที่มีความต่อเนื่องกัน รูปแบบสมการไฟไนต์เอลิเมนต์ของออยเลอเรียนแบบไม่คำนึงถึงผลของความยืดหยุ่นได้ถูกสร้างขึ้นและนำเสนอในส่วนแรกของบทความ โดยสมมุติว่าวัสดุภายใต้การเปลี่ยนแปลงรูปร่างมีพฤติกรรมที่เทียบเท่ากับของไหลแบบนอนนิวโตเนียนที่มีคุณสมบัติหนืด ไม่สามารถอัดตัวได้ เป็นเนื้อเดียวกันในเชิงมหภาคและไม่ขึ้นกับทิศทาง (ไอโซทรอปิค) เพื่อการทำแบบจำลองพฤติกรรม วิสโคพลาสติคของวัสดุภายใต้ขบวนการอัดขึ้นรูปด้วยเงื่อนไขที่ไม่มีการสูญเสียความร้อน (ไอโซเทอร์มัล) แบบจำลองของบิงแฮมได้ถูกเลือกเพื่อใช้อธิบายความสัมพันธ์คอนสทิทิวทีฟของวัสดุ ในงานวิจัยปัจจุบันเฉพาะกรณีสมมาตรในแนวแกนจะถูกพิจารณาและไฟไนต์เอลิเมนต์ของของแข็งแบบ 2 มิติจะถูกใช้เพื่อทำการจำลองขบวนการอัดขึ้นรูปทรงกระบอกแบบไปข้างหน้า ความเสียดทานที่ขอบข่ายระหว่างชิ้นงานกับส่วนของเครื่องมือขึ้นรูป อาทิเช่น พันช์ คอนเทนเนอร์ และ ดายน์ ไม่ถูกนำมาเพื่อพิจารณาในรูปแบบสมการปัจจุบัน ในขบวนการอัดขึ้นรูปทรงกระบอกแบบไปข้างหน้าอย่างสมบูรณ์ คุณภาพของผลิตภัณฑ์โดยปกติจะถูกผลิตในสภาวะค่อนข้างคงที่เมื่อวัสดุถูกอัดให้ไหลผ่านชุดแม่พิมพ์ เพื่อเป็นการนำเอาข้อได้เปรียบของการสำรวจชิ้นส่วนแปรรูป ณ สภาวะคงที่ การอ้างอิงด้วยระบบแกนตามออยเลอเรียนซึ่งกำหนดแกนโคออดิเนทให้คงที่ในบริเวณที่เนื้ออนุภาคของวัสดุไหลผ่านจะถูกประยุกต์ใช้ ด้วยคุณลักษณะเช่นนี้สามารถช่วยลดความต้องการของการทำแบบจำลองเชิงตัวเลขลงได้อย่างมาก ตัวอย่างเช่น การที่ต้องทำโครงตาข่ายซ้ำๆ ตลอดทั้งขบวนการเป็นสิ่งที่หลีกเลี่ยงไม่ได้

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Finite Element Modelling of Materials under an Axisymmetric Forward Bar Extrusion; Part I: Flow Formulation

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Abstract

The current research paper, dealing with the application of finite element (FE) technique to simulation of bulk forming processes, consists of a series of two parts. An Eulerian finite element flow formulation by assuming that materials under deformation behave identically to a viscous incompressible, macroscopically homogeneous and isotropic non-Newtonian fluid has been adopted and presented in the first part of paper. To model the viscoplastic behavior of a material under extrusion process with isothermal condition, the Bingham's model has been chosen to describe the constitutive relation of the material. In the present work, only an axisymmetric case is considered and 2-D solid finite elements are employed to simulate a forward bar extrusion. The boundary friction between the workpiece and tooling parts; i.e., punch, container and die does not take into consideration in the current formulation. In a complete forward bar extrusion process, quality products are usually produced in a fairly steady state as the material flows through the die. To take advantage of investigating the steady state forming part, an Eulerian description has to be employed which sets the coordinate axes fixed in space as a material particle passes through. With this manner, it can also help to reduce the demand of greatly numerical simulation; e.g. re-meshing throughout the process would be inevitable.

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1. Introduction

An understanding of the deformation of materials and the associated stresses and strains is crucial in order to determine the optimum means of manufacturing quality products safely, accurately, economically and efficiently. It is also essential in ensuring that products are able to withstand the normal loads to which they are to be subjected when in use. This is true whether the products are for aerospace, aircraft, automobile or other industrial applications.

Among numerous forming processes, the use of extrusion has been on the increase due to the development in the equipment and materials used. It produces constant section prismatic pieces with acceptable allowances and surface finishes [1]. The design of such processes has been in a trial-and-error fashion traditionally, which is in general costly and time consuming. An optimum design is unlikely to be achieved in this way. With the increasing capacity of computer modelling and simulation, it has been a major focus of research to model such forming processes accurately so that design can be made and optimised computationally. The economic benefit of achieving such a goal is obvious. However, there are gaps in the development of such analytical tools due to the complexity of the problem in nature, especially concerning simulation of the process of deformation.

It is obvious that the physical phenomena describing a forming operation are complex to express in a quantitative term [2]. They involve, in general, metal flow kinematics, friction at tool-workpiece interfaces, heat generation due to plastic deformation and friction and heat transfer within workpiece and between workpiece and tool, and the constitutive behaviour of material under process conditions. The finite element method (FEM) has been established as the most versatile simulation technique for the detailed study of the behaviour of materials in various forming processes [3]. Knowledge of the flow and stresses in forming operations is of great importance in assessing process parameters and predicting forming loads and the properties of the final product [4].

In analysing metal-forming processes, plastic strains normally outweigh elastic strains and idealisation of rigid-plastic or rigid-viscoplastic material behaviour is acceptable. The analysis based on this assumption is known as the *flow formulation* [5]. The use of rigid-plastic or rigid-viscoplastic material models in the flow formulation is usually effective and accurate for forming load, plastic strain and velocity distribution [6] for most metal-forming processes, although residual stresses and elastic spring-back effects are not predicted.

In the subsequent section of this paper, the Eulerian FE flow formulation that is derived by assuming that material behaves like an *incompressible non-Newtonian viscous fluid* [7] will be expressed in detail.

2. Flow Formulation for Axisymmetric Extrusion Process

The extrusion process can be considered to consist of large deformation of a rigid-plastic or rigid-viscoplastic material. This can be described as a flow of a *viscous incompressible isotropic non-Newtonian fluid* whose viscosity is a function of the strain, strain rate and temperature, in general [8].

2.1 Lagrangian and Eulerian Description

For the description of the motion of a body in the FE analysis, there are three methods of formulation: the *Lagrangian description*, the *Eulerian description,* and the *updated Lagrangian description* [9].

(1) In the *Lagrangian (referential) description,* the motion of a body is described through the motion of every material point in the body. It describes what happens to a material point at a time, wherever it is in space. Assume the current coordinates of a material point are **X** relative to a fixed coordinate system in space. **X** is in general a function time *t*, i.e. $\mathbf{X}=\mathbf{X}(t)$. The material point *X* can be labelled by its initial coordinates, i.e. $X=\mathbf{X}(0)$. Obviously, *X* does not change during the deformation. All of the field variables involved in the motion, such as displacements, velocities and accelerations, can be defined as functions of *X* and *t*.

(2) In the *Eulerian (spatial) description,* the motion of a body is defined as the motion at every point in space. It describes what is going on at a point in space at a time, whichever material point happens to be there. A point in space can be defined by its coordinates *X* in a fixed coordinate system. While a point in space does not change during the deformation, a material point occupies different point in space at different time. All of the field variables involved in the motion, such as displacements, velocities and accelerations, can be defined as functions of *X* and *t*.

(3) The *updated Lagrangian (relative) description* is a special type of Lagrangian description in which the reference configuration is a deformed state of the material at a given time t_0, i.e. $X=\mathbf{X}(t_0)$. It is often used to describe a motion in an incremental form so that in each increment the deformation is relatively small. The deformed configuration needs to be updated at the end of each increment.

With applications to extrusion problems in mind, a major difference between the Eulerian and Lagrangian formulation is a dominant part of an extrusion process can be described as a steady state problem in the former while the complete process must be described as a transient problem in the latter [10]. As mentioned above, it can be concluded that the nature of the Eulerian formulation is suitable for the study of flow problems. Therefore, the analysis of metal-forming processes, which operate under steady-state conditions such as extrusion, wire drawing, and rolling, is then facilitated by the use of fixed mesh FE analysis based on the developed Eulerian formulation [11].

2.2 Governing Equations for Axisymmetric Extrusion Problem

The *Navier-Stokes equations* govern an extensive range of fluid flow problems. These equations represent the conservation of linear momentum taking account of the constitutive relationship and the kinematics of the motion [12]. For the present application, only the case of viscous incompressible non-Newtonian fluid will be presented.

2.2.1 Constitutive relationship

By using *Perzyna's viscoplasticity model* [13] for a von *Mises* type of rigid-plastic or rigid-viscoplastic material, the constitutive relation between the real *(Cauchy)* stress $\boldsymbol{\sigma}$ and the strain rate $\dot{\boldsymbol{\varepsilon}}$ for a viscous incompressible isotropic non-Newtonian fluid in the cylindrical coordinate system (an axially symmetric case) is given in a matrix notation as [7-8]:

$$\boldsymbol{\sigma} = (\mu \mathbf{D}) \cdot \dot{\boldsymbol{\varepsilon}} - \mathbf{I} \cdot p \tag{1}$$

where $\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{\mathrm{r}} \\ \sigma_{\mathrm{z}} \\ \sigma_{\theta} \\ \sigma_{rz} \end{Bmatrix}$ is the Cauchy stress,

$\dot{\boldsymbol{\varepsilon}} = \begin{Bmatrix} \dot{\varepsilon}_{\mathrm{r}} \\ \dot{\varepsilon}_{\mathrm{z}} \\ \dot{\varepsilon}_{\theta} \\ 2\dot{\varepsilon}_{\mathrm{rz}} \end{Bmatrix}$ is the rate of strain,

$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a matrix matching tensorial to engineering strain rates,

$\mathbf{I} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$ is a vector whose function is equivalent to the *Kronecker delta* δ_{ij},

$$p = -\sigma_m = -\frac{\sigma_{ii}}{3} \tag{2}$$

is the pressure, which is equal to the negative value of the mean stress σ_m and μ is the non-linear viscosity which is in general a function of strain rate.

In the above, a dot represents the material derivative with respect to time and a repeated index implies summation.

From the above relationship between stresses and strain rates, it can be noted that all the shear components associated with the circumferential direction (hoop) vanish due to axisymmetry of the problem. As a result, only four components of the strain rate, and hence the stress, are involved in the present problem.

Alternatively, the constitutive equation, Eq. (1), can be rewritten in terms of the deviatoric stress $\mathbf{S}$ as the function of the strain rate $\dot{\boldsymbol{\varepsilon}}$, as [7]:

$$\mathbf{S} = \boldsymbol{\sigma} + \mathbf{I} \cdot p = (\mu \mathbf{D}) \cdot \dot{\boldsymbol{\varepsilon}} \qquad \textbf{(3)}$$

where $\mathbf{S} = \begin{Bmatrix} S_r \\ S_z \\ S_\theta \\ S_{rz} \end{Bmatrix}$ is the deviatoric stress.

On the other hand, according to the Levy-Mises flow law [6][14],

$$\mathbf{S} = \left(\frac{2\bar{\sigma}}{3\dot{\bar{\varepsilon}}} \right) \cdot \dot{\boldsymbol{\varepsilon}} \qquad \textbf{(4)}$$

where $\dot{\bar{\varepsilon}}$ is the Mises equivalent strain rate, given as:

$$\dot{\bar{\varepsilon}}^2 = \frac{2}{3} \cdot (\dot{\boldsymbol{\varepsilon}} \circ \dot{\boldsymbol{\varepsilon}}) \qquad \textbf{(5)}$$

and $\bar{\sigma}$ is the Mises equivalent stress, defined as:

$$\bar{\sigma}^2 = \frac{3}{2} \cdot (\mathbf{S} \circ \mathbf{S}). \qquad \textbf{(6)}$$

To establish the plastic flow, $\bar{\sigma}$ must be equal to the flow stress of the material at the current state of plastic deformation.

By comparison between Eq. (3) and Eq. (4), it can be found that

$$\mu = \frac{\bar{\sigma}}{3\dot{\bar{\varepsilon}}} \quad \textbf{(7a)}$$

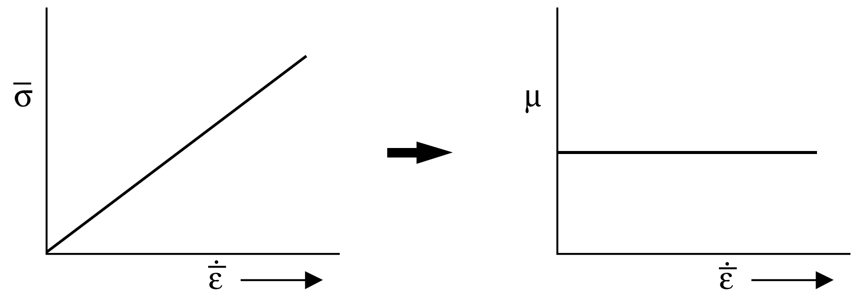
or $$\bar{\sigma} = 3\mu\dot{\bar{\varepsilon}}. \quad \textbf{(7b)}$$

From Eq. (7b), the viscosity can be obtained from the flow stress and the equivalent strain rate. A few typical such relationships for various materials have been schematically shown in Figure 1. The viscosity-effective strain rate dependence of many non-Newtonian materials such as polymers or melt metals can be written in form of the exponential relation for isothermal conditions, which is the so-called *Ostwald de Waele law,* as [6][10]:

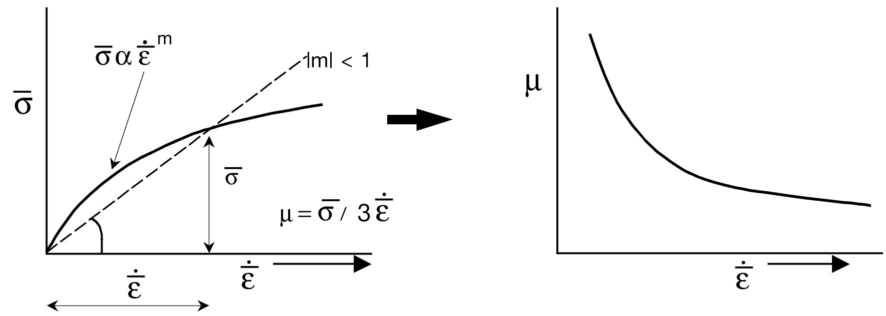
$$\mu = \mu_o \cdot \dot{\bar{\varepsilon}}^{(m-1)} \quad \textbf{(8a)}$$

with $$\mu_o = \mu_o(T, \bar{\varepsilon}) \quad \textbf{(8b)}$$

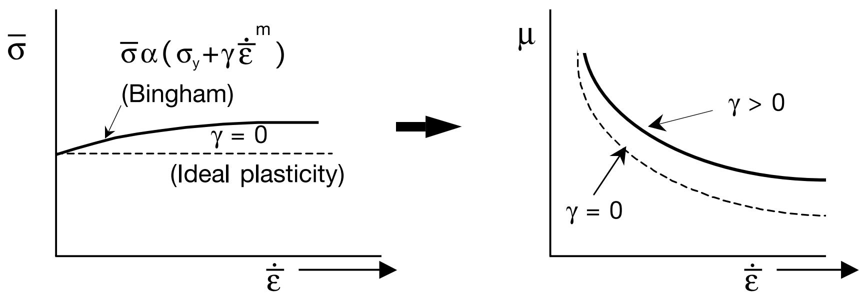
where T is the temperature and $\bar{\varepsilon}$ means the total strain invariant. The absolute magnitude of m value should be less than 1.



(a) linear Newtonian fluid



(b) non-Newtonian polymers



(c) non-Newtonian perfect plastic

and viscoplastic metals

Fig. 1 Flow stress and viscosity versus effective strain rate for various material models

In a similar manner the viscosity law can be found for viscoplastic behaviour of an incompressible kind, for instance, the *Bingham fluid* [15]. In this case, the non-linear viscosity is related to the yield stress and the equivalent strain rate by the following equation [6][10]:

$$\mu = \frac{\sigma_y + \gamma \cdot \dot{\bar{\varepsilon}}^m}{3\dot{\bar{\varepsilon}}} \quad \textbf{(9a)}$$

where σ_y is the uniaxial yield stress which, for strain hardening materials, is a function of the total accumulated strain and temperature in general, then

$$\sigma_y = \sigma_y\,(T, \bar{\varepsilon}) \quad \textbf{(9b)}$$

and γ is the fluidity parameter which its magnitude is equal (perfect plastic) or greater than zero and m is the strain rate sensitivity exponent.

However, when the viscous effects become negligible, the fluidity coefficient $\gamma = 0$, the viscosity in Eq. (9a) is simplified to that for perfect plasticity:

$$\mu = \frac{\sigma_y}{3\dot{\bar{\varepsilon}}}. \quad \textbf{(10)}$$

2.2.2 Kinematics equation

The constitutive equations, Eqs. (1) to (3), define the Cauchy stress $\boldsymbol{\sigma}$ or the deviatoric stress $\mathbf{S}$ as functions of the strain rate $\dot{\boldsymbol{\varepsilon}}$ for the flow of a fluid. In turn, this strain rate can be defined by the spatial derivatives of the velocity $\mathbf{u}$ according to the kinematics [12] as:

$$\dot{\boldsymbol{\varepsilon}} = \Re \cdot \mathbf{u} \quad \textbf{(11)}$$

where $\Re = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}$ is the matrix operator and $\mathbf{u} = \begin{Bmatrix} u_r \\ u_z \end{Bmatrix}$ is the velocity vector.

2.2.3 Continuity equation (incompressibility condition)

In general, materials are taken to be incompressible viscous non-Newtonian fluid during extrusion processes. Consequently, the continuity equation or incompressibility constraint of this fluid can be expressed in a matrix notation as [12]:

$$\mathbf{I}^T \cdot \dot{\varepsilon} = \mathbf{I}^T \cdot \Re \cdot \mathbf{u} = 0 \ . \tag{12}$$

2.2.4 Conservation of linear momentum

The conservation of linear momentum in fluid mechanics is normally given in the form of *Cauchy's equation of motion*, which is derived from Newton's second law of motion [12][16]. Physically, it illustrates a unique balance of body and surface forces with inertia forces. If the body forces do not nullify the surface forces a nonzero inertia force will cause the fluid acceleration [16]. The expression of Cauchy's equation of motion for viscous incompressible fluid, for which the constitutive equation (the Cauchy stress-rate of strain relationship) has been given in Eq. (1), can be obtained as [6]:

$$\Theta^T \cdot \boldsymbol{\sigma} + \mathbf{b} = \rho \cdot \mathbf{a} \tag{13}$$

where $\mathbf{\Theta} = \begin{bmatrix} \frac{\partial}{\partial r} + \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ -\frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} + \frac{1}{r} \end{bmatrix}$

$\mathbf{b}$ is the body force vector, ρ is the density of the material/fluid and $\mathbf{a}$ represents the acceleration vector.

In the Eulerian description, the acceleration term in Eq. (13) should be derived using the *substantive derivative* (or the *derivative following the fluid,* or the *material derivative*) operator $\frac{\mathrm{D}}{\mathrm{D}t}$ to distinguish from the simple time derivative $\frac{d}{dt}$ [12]. This derivative can be expressed as:

$$a_i = \left(\frac{\mathrm{D}}{\mathrm{D}t}\right)\cdot u_i = \dot{u}_i + u_j \cdot u_{i,j} \tag{14a}$$

In the above, a comma means a partial derivative with respect to a coordinate. Hence

$$\mathbf{a} = \dot{\mathbf{u}} + \left(\nabla\cdot\mathbf{u}^T\right)^T\cdot\mathbf{u} \tag{14b}$$

where $\nabla = \mathbf{\Theta}^T\mathbf{I} = \begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix}$.

In Eqs. (14a, b), the first term gives the local rate of change $\frac{\partial}{\partial t}$ and the second part shows the convective rate of change following the path of fluid particle, respectively. The convective rate of change represents the change in the continuum variable caused by the convection (spatial change) of a fluid particle from one location to another.

Substituting Eq. (1), Eq. (11) and Eq. (14b) into Eq. (13),

$$\mathbf{\Theta}^T\cdot(\mu\mathbf{D})\cdot\Re\cdot\mathbf{u} - \nabla\cdot p + \mathbf{b} = \rho\cdot[\dot{\mathbf{u}} + (\nabla\cdot\mathbf{u}^T)^T\cdot\mathbf{u}]. \tag{15}$$

The equations presented in this section are all expressed in the matrix form.

3. Finite Element Discretisation of Plastic or Viscoplastic Incompressible Flow Problems

In Eq. (15), there are three independent field variables (unknowns), two velocity components, u_r and u_z, and one pressure parameter *p*. The *Galerkin Finite Element Method* (GFEM) can be utilised for establishing the finite element formulation by discretisation of this flow equation [17]. With the GFEM, the *weighting functions* are chosen to be identical to the shape functions and the *residual* is defined as the error when the velocity field as in Eq. (15) is discretised [18].

For an 8-node quadrilateral element, the velocity field in the element can be discretised to nodal velocities $\mathbf{v}_i$ using the shape functions $\mathbf{N}_{ui}$ as follows [18]:

$$\mathbf{u} = \sum \mathbf{N}_{ui} \cdot \mathbf{v}_i = \mathbf{N}_u \cdot \mathbf{v} \tag{16}$$

where $\mathbf{v}_i = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$,

$$\mathbf{v} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & - & - & - & - & - & - & - & - & - & - & u_8 & v_8 \end{bmatrix}^T \tag{17a}$$

and $$\mathbf{N}_u = \begin{bmatrix} \mathbf{N}_{u1} & 0 & \mathbf{N}_{u2} & 0 & - & - & - & - & - & - & \mathbf{N}_{u8} & 0 \\ 0 & \mathbf{N}_{u1} & 0 & \mathbf{N}_{u2} & - & - & - & - & - & - & 0 & \mathbf{N}_{u8} \end{bmatrix}. \tag{17b}$$

All the sixteen nodal velocities have been collected into a nodal velocity vector $\mathbf{v}$ in Eq. (16). $\mathbf{N}_u$ is a 2x16 matrix of shape functions corresponding to the nodes involved in this element. The components of this matrix are a function of the coordinates. The expressions for these shape functions can be found in References [17] - [18].

In order to eliminate the pressure parameter from the computation [19], the *penalty constraint approach* is employed. With this, the negative value of an extra term $\frac{p}{\lambda}$ is added to the right-hand side of the continuity (incompressibility) equation, Eq. (12), as [6][10]:

$$\mathbf{I}^T \cdot \dot{\boldsymbol{\varepsilon}} = \mathbf{I}^T \cdot \Re \cdot \mathbf{u} = -\frac{p}{\lambda} \tag{18a}$$

or $$p = -\lambda \cdot \mathbf{I}^T \cdot \Re \cdot \mathbf{u} \tag{18b}$$

where λ is the penalty constant, which is normally given a very large value. Obviously, the continuity condition is satisfied exactly when λ approaches infinity but with a large value the condition is satisfied approximately. For a viscous flow, this penalty constant is often chosen according to the fluid viscosity as [6][10]:

$$\lambda = \left(10^7 - 10^{10}\right) \cdot \mu \ .$$

Substituting Eq. (16) and Eq. (18b) into Eq. (15) and applying the GFEM, the conservation of linear momentum (equilibrium equation) for an element is given as:

$$\int_{\Omega} \mathbf{W}_u^{\,T} \cdot \begin{bmatrix} \Theta^T \cdot (\mu \mathbf{D}) \cdot \Re \cdot \mathbf{N}_u \cdot \mathbf{v} + \nabla \cdot \left(\lambda \cdot \mathbf{I}^T \cdot \Re \cdot \mathbf{N}_u \cdot \mathbf{v}\right) + \mathbf{b} - \\ \rho \cdot \left\{ \mathbf{N}_u \cdot \dot{\mathbf{v}} + \left\{\nabla \cdot (\mathbf{N}_u \cdot \mathbf{v})^T\right\}^T \cdot \mathbf{N}_u \cdot \mathbf{v} \right\} \end{bmatrix} \cdot d\Omega = 0 \tag{19}$$

where $\mathbf{W}_u$ is the weighting function for velocity field.

With the weighting function chosen as the shape function, the above equation can be rewritten as:

$$\begin{bmatrix} \left[\int_{\Omega} \left\{ \mathbf{N}_u^{\,T} \cdot \mathbf{\Theta}^T \cdot (\mu \mathbf{D}) \cdot \Re \cdot \mathbf{N}_u \right\} \cdot d\Omega \right] \cdot \mathbf{v} + \left[\int_{\Omega} \left\{ \mathbf{N}_u^{\,T} \cdot \nabla \cdot \left(\lambda \cdot \mathbf{I}^T \cdot \Re \cdot \mathbf{N}_u\right) \right\} \cdot d\Omega \right] \cdot \mathbf{v} - \\ \left[\int_{\Omega} \left\{ \rho \cdot \mathbf{N}_u^{\,T} \cdot \left\{\nabla \cdot (\mathbf{N}_u \cdot \mathbf{v})^T\right\}^T \cdot \mathbf{N}_u \right\} \cdot d\Omega \right] \cdot \mathbf{v} - \left[\int_{\Omega} \left(\rho \cdot \mathbf{N}_u^{\,T} \cdot \mathbf{N}_u\right) \cdot d\Omega \right] \cdot \dot{\mathbf{v}} + \\ \int_{\Omega} \left(\mathbf{N}_u^{\,T} \cdot \mathbf{b}\right) \cdot d\Omega \end{bmatrix} = 0 \tag{20}$$

By application of the *Green's theorem* to the first and second terms of Eq. (20), one obtains:

$$\begin{aligned} &\left[\int_{\Omega} \left\{ \mathbf{B}^T \cdot (\mu \mathbf{D}) \cdot \mathbf{B} \right\} \cdot d\Omega \right] \cdot \mathbf{v} + \left[\int_{\Omega} \left\{ \mathbf{B}^T \cdot (\mathbf{I} \cdot \lambda \cdot \mathbf{I}^T) \cdot \mathbf{B} \right\} \cdot d\Omega \right] \cdot \mathbf{v} + \\ &\left[\int_{\Omega} \left\{ \rho \cdot \mathbf{N}_u^{\,T} \cdot \left\{\nabla \cdot (\mathbf{N}_u \cdot \mathbf{v})^T\right\}^T \cdot \mathbf{N}_u \right\} \cdot d\Omega \right] \cdot \mathbf{v} + \left[\int_{\Omega} (\rho \cdot \mathbf{N}_u^{\,T} \cdot \mathbf{N}_u) \cdot d\Omega \right] \cdot \dot{\mathbf{v}} \\ &= \int_{\Omega} (\mathbf{N}_u^{\,T} \cdot \mathbf{b}) \cdot d\Omega + \int_{\Gamma_t} (\mathbf{N}_u^{\,T} \cdot \mathbf{t}) \cdot d\Gamma \end{aligned} \tag{21}$$

where $\mathrm{t} = \begin{Bmatrix} \mathrm{t}_r \\ \mathrm{t}_z \end{Bmatrix}$ is the vector of distributed traction on the surface Γ_t, as will be shown in detail in subsequent section, and $\mathbf{B}$ is often called the kinematical matrix for the element, which is a 4x16 matrix and comprises of the derivatives of the shape functions with respect to the coordinates. It relates the nodal velocities in the element to the velocity field in the element. It can be given as follows

$$\mathbf{B} = \Re \cdot \mathbf{N}_u = \begin{bmatrix} \frac{\partial \mathbf{N}_{u1}}{\partial r} & 0 & - & - & - & - & - & - & \frac{\partial \mathbf{N}_{u8}}{\partial r} & 0 \\ 0 & \frac{\partial \mathbf{N}_{u1}}{\partial z} & - & - & - & - & - & - & 0 & \frac{\partial \mathbf{N}_{u8}}{\partial z} \\ \frac{\mathbf{N}_{u1}}{r} & 0 & - & - & - & - & - & - & \frac{\mathbf{N}_{u8}}{r} & 0 \\ \frac{\partial \mathbf{N}_{u1}}{\partial z} & \frac{\partial \mathbf{N}_{u1}}{\partial r} & - & - & - & - & - & - & \frac{\partial \mathbf{N}_{u8}}{\partial z} & \frac{\partial \mathbf{N}_{u8}}{\partial r} \end{bmatrix} . \quad \textbf{(22)}$$

Symbols of $\int_\Omega d\Omega$ and $\int_\Gamma d\Gamma$ in Eq. (21) represent integrals over the volume and surface of the domain, respectively. In Eq. (21), the first three terms of volume integrals on the left-hand side produces the element stiffness matrices, denoted as $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$, respectively. The last term on the left-hand side gives the mass matrix **M**, which is always involved in a transient problem [10]. It is obvious that matrices $\mathbf{K}_1$ and $\mathbf{K}_2$ are symmetric, whereas $\mathbf{K}_3$, arising from the convective effects, is asymmetric in general. The first term on the right-hand side of Eq. (21) is the element force vector due to body forces and the second term due to the distributed surface traction.

For the axisymmetric problems, due to the fact that all the field variables such as stresses and strains do not vary in the circumferential direction of a cylindrical coordinate system, the volume and surface integrals become

$$\int_\Omega d\Omega \equiv 2\pi r \int dA \equiv 2\pi r \iint drdz \quad \textbf{(23a)}$$

and

$$\int_\Gamma d\Gamma \equiv 2\pi r \int ds \quad \textbf{(23b)}$$

where r and z are the radial and longitudinal coordinates, respectively, and *s* is the arc length along the surface involve in this element.

The final flow equation for the element can be obtained by implementing Eqs. (23a, b) to Eq. (21), leading to:

$$\mathbf{M} \cdot \dot{\mathbf{v}} + (\mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3) \cdot \mathbf{v} = \mathbf{f} \tag{24}$$

where $$\mathbf{M} = \iint (\rho \cdot \mathbf{N}_u{}^T \cdot \mathbf{N}_u) \cdot 2\pi r \cdot drdz \tag{25a}$$

$$\mathbf{K}_1 = \iint \{\mathbf{B}^T \cdot (\mu \mathbf{D}) \cdot \mathbf{B}\} \cdot 2\pi r \cdot drdz \tag{25b}$$

$$\mathbf{K}_2 = \iint \{\mathbf{B}^T \cdot (\mathbf{I} \cdot \lambda \cdot \mathbf{I}^T) \cdot \mathbf{B}\} \cdot 2\pi r \cdot drdz \tag{25c}$$

$$\mathbf{K}_3 = \iint [\rho \cdot \mathbf{N}_u{}^T \cdot \{\nabla \cdot (\mathbf{N}_u \cdot \mathbf{v})^T\}^T \cdot \mathbf{N}_u] \cdot 2\pi r \cdot drdz \tag{25d}$$

$$\mathbf{f} = \iint (\mathbf{N}_u{}^T \cdot \mathbf{b}) \cdot 2\pi r \cdot drdz + \int (\mathbf{N}_u{}^T \cdot \mathbf{t}) \cdot 2\pi r \cdot ds\,. \tag{25e}$$

4. Conclusions

A FE viscoplastic flow formulation for axisymmetric problems has been successfully established in the present work using an Eulerian description, assuming the material behaves like an incompressible isotropic viscous non-Newtonian fluid. This 2-D FE flow formulation adopted will be employed for modelling the deformation behaviour of materials in forward bar extrusion with an isothermal condition. The problem is highly non-linear in nature because of the material behaviour. For the first attempt, the boundary friction and heat transfer between the workpiece-tooling parts such as punch, container, and die, interfaces do not take into account within the current formulation. Discussions have been made on the advantages of applying an Eulerian description in conjunction with the Bingham's viscoplastic constitutive relationship for modelling materials under such forming process.

In general, most important features of the forward bar extrusions are present in the steady state period and can therefore be explored in such a manner. Investigating the steady state part of the complete extrusion process alone helps to reduce the demand of simulation greatly as, otherwise, simulating such forming process as a transient problem would be highly demanding numerically, e.g. re-meshing throughout the process would be inevitable. To take advantage of the steady state forming part, an Eulerian description has to be employed.

In conclusion, through the present project, it has been made possible to simulate the forward bar extrusion problem of materials. The formulation developed will provide a useful mathematical tool for modelling extrusion processes in future.

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