การจัดการกับเงื่อนไขขอบเขตที่ไม่ต่อเนื่อง โดยอาศัยความเป็นไปได้อย่างเดียวของฟลักซ์ความร้อน ในการใช้วิธีบาวน์ดะรีเอลิเมนต์แก้ปัญหาการนำความร้อน

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บทคัดย่อ

ในการแก้ปัญหาการนำความร้อนด้วยวิธีบาวน์ดะรีเอลิเมนต์มีบ่อยครั้งที่ต้องพบกับฟลักซ์ความร้อนในแนวตั้งฉากกับขอบเขตที่มีค่าไม่ต่อเนื่อง ความยุ่งยากอาจเกิดขึ้นในกรณีที่ต้องหาค่าฟลักซ์ความร้อนในแนวตั้งฉากกับขอบเขตที่ขอบหรือมุมของขอบเขต เพราะระบบสมการบาวน์ดะรีเอลิเมนต์อาจมีสภาวะล้มเหลว บทความนี้นำเสนอเทคนิคที่ใช้ความเป็นไปได้อย่างเดียวของฟลักซ์ความร้อนที่ขอบหรือมุมในการสร้างสมการเพิ่มเติม ซึ่งทำให้ได้ระบบสมการบาวน์ดะรีเอลิเมนต์ที่มีสภาวะใช้งานได้ มีการนำเทคนิคนี้ไปใช้อธิบายปัญหาสองมิติและปัญหาสามมิติได้โดยกระจ่าง การใช้เทคนิคนี้กับปัญหาตัวอย่างแสดงให้เห็นว่าเทคนิคนี้ง่ายและมีประสิทธิผล

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Handling Discontinuous Boundary Condition by Using the Uniqueness of Heat Flux in the Boundary Element Method of Solving Heat Conduction Problem

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Abstract

In solving heat conduction problems using the boundary element method, dis-continuous normal heat flux at boundary is usually encountered. Difficulty may arise when normal heat flux is to be determined at edges or corners where there is discontinuity because the system of boundary element equations may be ill-conditioned. This paper presents a technique that makes use of the uniqueness of heat flux vector at edges and corners to generate additional equations that will give a well-conditioned system of boundary element equations. Implementation is described for two-dimensional problems and three-dimensional problems. Examples show that this technique is simple and effective.

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1. Introduction

Although temperature and heat flux are continuous functions, normal heat flux at edges or corners may be discontinuous because of discontinuous normal vectors. The boundary element method for solving heat conduction problem requires a numerical technique to deal with discontinuities in normal heat flux at edges and corners. Several techniques have been proposed. The use of nonconforming or discontinuous elements avoids such difficulties altogether by not placing a boundary node at an edge or a corner [1]. However, conforming elements are usually preferred because they have the higher potential to produce accurate solutions [2].

When conforming elements are used, heat flux must be allowed to be multiple-valued at an edge or a corner. This is accomplished by placing multiple nodes there. If the boundary condition at only one side of the edge or the corner is of Neumann type (i.e. normal heat flux is specified), no problem arises. If not, however, the boundary element method will yield distorted solution that has nearly equal values of secondary variables at different sides of the edge or the corner. This is due to the fact that collocation at the multiple nodes at the edge or the corner yield similar algebraic equations, leading to a high condition number in the system of boundary element equations. By supplying auxiliary equations, the tendency to yield nearly equal values of secondary variables at the edge or the corner may be circumvented.

Auxiliary equations may be obtained by collocating at additional points either outside the domain or on the boundary [3],[4]. This method is quite simple, and applicable to any problem. Previous studies have shown the effectiveness of this method. However, the author has been unsuccessful in applying this method to the three-dimensional heat conduction problem. This may be due to the fact the boundary element method can produce accurate solution at boundary nodes and interior nodes that are not too close to the boundary.

Another way to derive auxiliary equations is by making use of the fact that heat flux must be unique everywhere on the boundary. Chan and Chandra [2] suggested an algorithm based on this idea to handle corners in two-dimensional problems. This paper presents a technique based on this idea also. However, this technique is generalized to three-dimensional problems. In the following sections, brief description of the boundary element formulation for heat conduction problems is described. Subsequently, implementation is described for two-dimensional problems that use quadratic element and three-dimensional problems that use six-node triangular element or eight-node quadrilateral element. Finally, examples are given to show the effectiveness of this technique.

2. General Boundary Element Formulation

For a heat conduction problem described by the Laplace's equation:

$$\nabla^2 T(x,y) \quad = \quad 0 \tag{1}$$

where the Laplace operator is $\partial^2/\partial x^2 + \partial^2/\partial y^2$ for two-dimensional problems or $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ for three-dimensional problems. The corresponding boundary integral equation is [5]

$$c\left(\vec{\xi}\right)T\left(\vec{\xi}\right) \quad = \quad \int_\Gamma \frac{q(\vec{r})}{\kappa} G\left(\vec{\xi},\vec{r}\right) d\Gamma - \int_\Gamma T(\vec{r}) \frac{\partial G\left(\vec{\xi},\vec{r}\right)}{\partial n} d\Gamma \tag{2}$$

where κ is thermal conductivity, c is coefficient that depends on the location of ξ, T is temperature, q is normal heat flux, G is the fundamental solution, and n is the coordinate normal to the boundary. Note that the boundary is a closed curve for a two-dimensional problem, or a closed area for a three-dimensional problem. If the boundary is discretized, appropriate interpolation functions are applied to Eq. 2, and numerical integration is preformed, the result is a system of boundary element equations [5]:

$$\sum_{j=1}^{N} A_{ij} q_j - \sum_{j=1}^{N} B_{ij} T_j \quad = \quad 0 \qquad (i=1,2, ..., N) \tag{3}$$

where A_{ij} and B_{ij} are coefficients that result from numerical integration of integrals in Eq. 2, and N is the number of boundary nodes. In order to allow for discontinuous normal heat flux at edges and corners, multiple nodes are placed at edges and corners. This means that this system of algebraic equations is ill-conditioned because if nodes k and l are at the same coordinates, A_{kj} will be nearly equal to A_{lj}, and B_{kj} will be nearly equal to B_{lj}. The solution to Eq. 3 may still be possible, but q_k and q_l will have almost the same value although they should be different. Consequently, the boundary element solution to this heat conduction problem may be inaccurate if nothing is done to improve the condition number of Eq. 3.

3. Two-dimensional Problem

One way to make Eq. 3 well-conditioned is to replace one or more boundary element equations at edges and corners with equations that are not obtained from Eq. 2. The use of unique heat flux can yield these equations. Consider a corner of a two-dimensional domain where elements (i) and (ii) meet as shown in Fig. 1. Notice that there are two nodes (3 and 4) at the corner, and normal heat flux q_3 at node 3 of element (i) is not equal to normal heat flux q_4 at node 4 of element (ii). Even though normal heat flux is discontinuous at the corner, heat flux vector of which normal and tangential components at node 3 of element (i) are q_3 and q_{3t}, must be unique. Therefore, the vector sum of q_3 and q_{3t} must be equal to the vector sum of q_4 and q_{4t}, which denote the normal and tangential components at node 4 of element (ii). The following relation can be written.

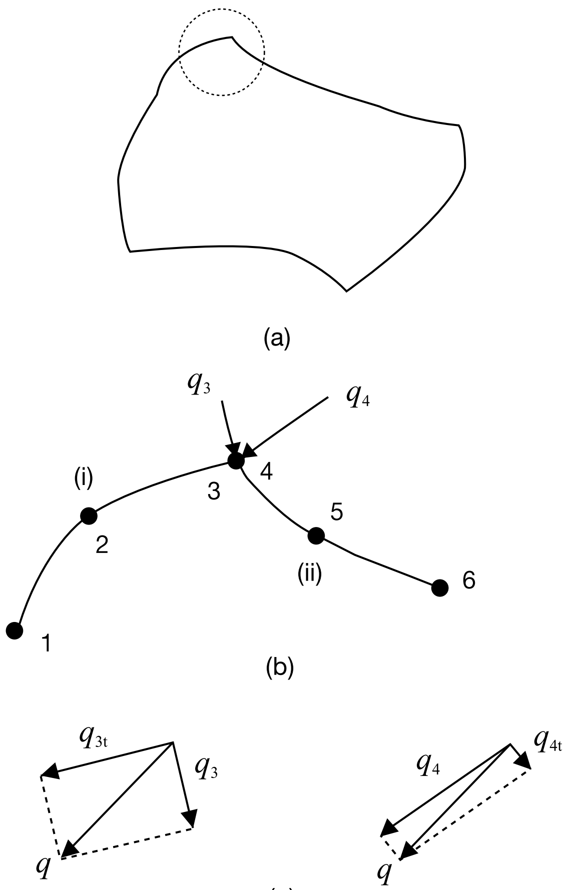

Fig. 1 (a) Domain and boundary of a two-dimensional heat conduction problem. The corner inside the dashed circle is where normal heat flux is discontinuous. (b) Boundary around the corner is magnified to show elements, nodes, and unequal nornal heat flux components. (c) Because nodes 3 and 4 have the same coordinates, heat flux vector q at these nodes must be equal even though $q_3 \neq q_4$.

$$q_4 = q_3\cos\alpha_1 + q_{3t}\cos\alpha_2 \quad (4)$$

where α_1 and α_2 are angle between q_4 and q_3 and angle between q_4 and q_{3t}, respectively. In implementing this technique, the boundary element equation at node 3 (i = 3 in Eq. 3) can be obtained from Eq. 2 as usual. But the boundary element equation at node 4 is replaced by Eq. 4 with q_{3t} related to temperatures at nearby nodes by the Fourier's law.

$$q_{3t} = \frac{-\kappa}{\sqrt{(dx/ds)^2 + (dy/ds)^2}}\frac{\partial T}{\partial s} \quad (5)$$

where all derivatives with respect to s must be evaluated at the value of s corresponding to node 3.

4. Three-dimensional Problem

In a three-dimensional problem, normal heat flux may be discontinuous at an edge or a corner, but heat flux must be unique at either location. There are 2 nodes at the edge or 3 nodes at the corner. Analogous technique can be used to derive one equation to replace a boundary element equation. In Eq. 3 in the case of the edge or 2 equations in the case of the corner. Fig. 2 shows the orientation of the normal heat flux (q_2) at one side of an edge with respect to the three components of the heat flux vector (q_1, q_{t1}, q_{t2}) at the other side of the edge. The relation between q_2 and the three heat flux components is similar to Eq. 4.

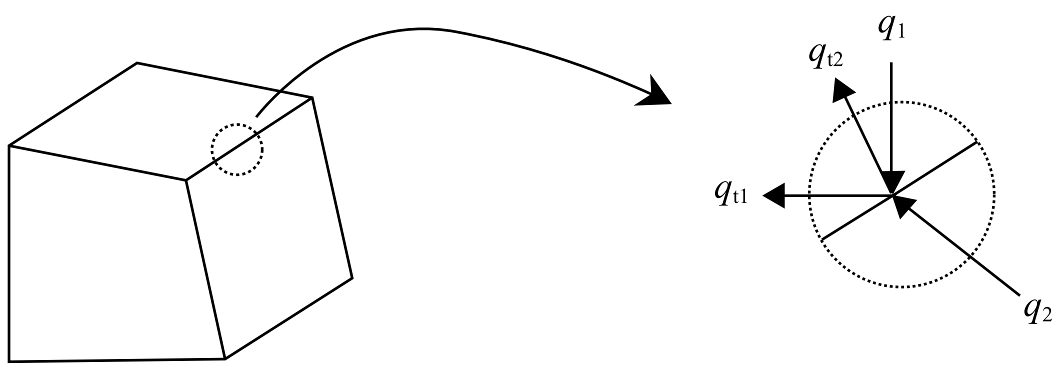

Fig. 2 Normal heat flux components q_1 and q_2 at the edge may not be equal. However, because heat flux at the edge is unique, q_2 can be related to q_1, q_{t1}, and q_{t2}.

$$q_2 = q_1\cos\alpha_1 + q_{t1}\cos\alpha_2 + q_{t2}\cos\alpha_3 \quad \textbf{(6)}$$

where α_1, is the angle between q_2 and q_1, α_2 is the angle between q_2 and q_{t1}, and α_3 is the angle between q_2 and q_{t2}. Heat flux components q_{t1} and q_{t2} are in the direction of s_1 and s_2, respectively, which are the parameters used to describe the surface boundary. Tangential components of heat flux at the edge are related to temperatures at nearby nodes by the Fourier's law.

$$q_{t1} = \frac{-\kappa}{\sqrt{(\partial x/\partial s_1)^2 + (\partial y/\partial s_1)^2 + (\partial z/\partial s_1)^2}}\frac{\partial T}{\partial s_1} \quad \textbf{(7)}$$

$$q_{t2} = \frac{-\kappa}{\sqrt{(\partial x/\partial s_2)^2 + (\partial y/\partial s_2)^2 + (\partial z/\partial s_2)^2}}\frac{\partial T}{\partial s_2} \quad \textbf{(8)}$$

where all derivatives with respect to s_1 and s_2 must be evaluated at the values of s_1 and s_2 corresponding to the node under consideration.

Fig. 3 shows the orientation of the normal heat flux q_2 in element (ii) and the normal heat flux q_3 in element (iii) at the corner where elements (i), (ii), and (iii) meet with respect to the three heat flux components (q_1, q_{t1}, q_{t2}) in element (i). Two equations analogous to Eq. 6 can be used to replace 2 boundary element equations in Eq. 3 that correspond to corner nodes in elements (ii) and (iii).

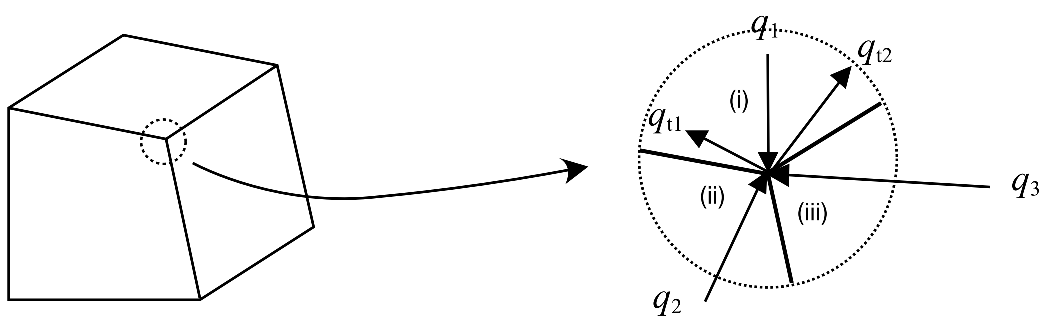

Fig. 3 Elements (i), (ii), and (iii) meet at the corner. Normal heat flux q_2 in element (ii) and normal heat flux q_3 in element (iii) at the corner can be related the three heat flux components in element (i).

5. Numerical Results

The proposed technique is tested by a two-dimensional example and a three-dimensional example. In both examples, the value of thermal conductivity is unity. The two-dimensional example is a heat conduction problem in a square of unit length as shown in Fig. 4. Let temperature distribution be described by

$$T(x,y) = \cos(x)\cosh(y) - \sin(x)\sinh(y) \tag{9}$$

Now suppose that the exact temperature distribution is unknown, but boundary conditions are known. Let temperature be specified on two sides of the square as shown in Fig. 4, and normal heat flux be specified on the other two sides. The boundary element method will be used to calculate normal heat flux.

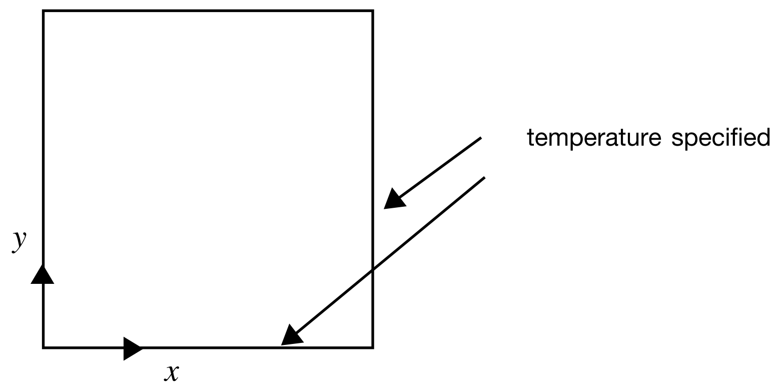

Fig. 4 The two-dimensional problem is a heat conduction problem within a square of unit length. Temperature is specified on two sides of the square, whereas heat flux is specified on the other two sides.

For the three-dimensional example, consider heat conduction in a cube of unit length shown in Fig. 5. Let temperature distribution be described by

$$T(x,y,z) = \cos(x)\cosh(y) - \sin(x)\sinh(z) \tag{10}$$

Now suppose that the exact temperature distribution is unknown, but boundary conditions are known. Let temperature be specified on three sides of the cube as shown in Fig. 5, and normal heat flux be specified on the other three sides. Again, the boundary element method will be used to calculate normal heat flux.

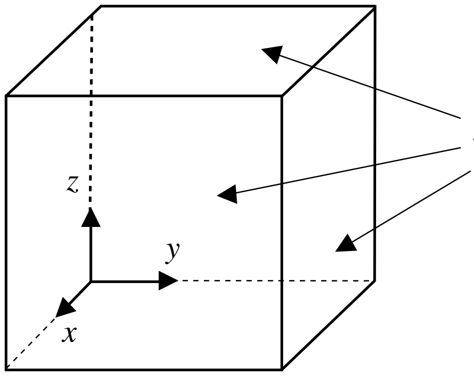

Fig. 5 The sample three-dimensional problem is a heat conduction problem within a cube of unit length. Temperature is specified on three sides of the cube, whereas heat flux is specified on the other three sides.

For both examples, the measure of accuracy is indicated by the L_2 norm of the difference between the exact heat flux (q_{exact}) and reproduced heat flux (q) on the boundary where temperature is specified:

$$\varepsilon \quad = \quad \left[\frac{1}{M} \sum_{i=1}^{M} (q_i - q_{i,exact})^2 \right]^{1/2} \tag{11}$$

where M is the number of nodes on the boundary where temperature is specified. Exact heat flux is obtained from the Fourier's law

$$q \quad = \quad \kappa \frac{\partial T}{\partial n} \tag{12}$$

with T is given in Eq. 10, and the reproduced heat flux is obtained from the boundary element method.

Table 1 Comparison of ε of boundary element solutions of the two-dimensional sample problem with and without using the proposed technique. The boundary is divided into equal quadratic elements.

M	Using the proposed technique	Not using the proposed technique
22	8.47×10^{-4}	0.255
42	2.05×10^{-4}	0.184
62	1.95×10^{-4}	0.152

Table 2 Comparison of ε of boundary element solutions of the three-dimensional sample problem with and without using the proposed technique. The boundary is divided into equal six-node triangular elements.

M	Using the proposed technique	Not using the proposed technique
75	2.54×10^{-2}	4.40
147	1.72×10^{-2}	0.429
243	1.45×10^{-2}	0.423

Table 3 Comparison of ε of boundary element solutions of the three-dimensional sample problem with and without using the proposed technique. The boundary is divided into equal eight-node quadrilateral elements.

M	Using the proposed technique	Not using the proposed technique
63	2.10×10^{-2}	0.545
120	9.47×10^{-3}	0.471
195	5.62×10^{-3}	0.422

Tables 1 - 3 compare ε of boundary element solutions of the sample two-dimensional and three-dimensional sample problems with and without using the proposed technique. (The solutions obtained without using the proposed technique are solutions to the system of algebraic equations in Eq. 3 without any special treatment to deal with discontinuous normal heat flux.) Each solution is obtained by dividing the boundary into equal elements so that the number of nodes on part of the boundary where temperature is specified is $\boldsymbol{M}$. It can be seen that boundary element solutions without using the proposed technique yield very high values ε. This is so because the boundary element method will make heat flux continuous at edges and corners, which will pollute results away from edges and corners. By contrast, boundary element solutions obtained by using the proposed technique yield low values ε, which monotonically decrease with increasing $\boldsymbol{M}$.

6. Conclusions

A numerical technique to resolve difficulties caused by discontinuous boundary conditions in the boundary element method is presented in this paper. In this technique, normal heat flux at any edge node or a corner node of a quadratic element, a six-node triangular element or an eight-node quadrilateral element is related to normal heat flux and derivatives of temperature at an adjacent element on the other side of the edge or corner. The technique simple to implement, and effective in solving sample problems.

7. References

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