การเปรียบเทียบวิธีเชิงตัวเลขในการแก้ปัญหา
การพาและการแพร่กระจาย

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บทคัดย่อ

ปัญหาการพาและการแพร่กระจายเป็นความท้าทายสำหรับวิธีเชิงตัวเลข แต่ก็มีวิธีที่สามารถแก้ปัญหานี้ได้อย่างมีประสิทธิภาพในกรณีที่ปัญหานี้เป็นปัญหาเอกพันธ์ วิธีดังกล่าวสามวิธีได้แก่ วิธีบาวน์ดะรีเอลิเมนต์ (BEM) วิธีบาวน์ดะรีน็อต (BKM) และวิธีผลเฉลยหลักมูล (MFS) วิธีเหล่านี้ต้องการเพียงบัพหรือเมซที่ขอบเขตของโดเมนของปัญหา มิใช่บัพหรือเมซภายในโดเมนดังเช่นที่ต้องการโดยวิธีไฟไนต์ดิฟเฟอเรนซ์และวิธีไฟไนต์เอลิเมนต์ บทความนี้เปรียบเทียบสมรรถนะของทั้งสามวิธีในการแก้ปัญหาการพาและการแพร่กระจายสองมิติ ผลการเปรียบเทียบจากปัญหาตัวอย่างแสดงให้เห็นว่า BKM และ MFS ให้ผลเฉลยที่แม่นยำกว่าในกรณีที่ผลเฉลยนั้นเป็นฟังก์ชันที่ค่อนข้างปรับเรียบ แต่ล้มเหลวถ้าผลเฉลยเป็นฟังก์ชันที่แปรผันอย่างรวดเร็ว ในทางตรงกันข้าม BEM สามารถให้ผลเฉลยแบบต่างๆ ได้และผลเฉลยที่ได้มีลักษณะลู่เข้า

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Comparison of Numerical Methods for Solving the Convective-Diffusive Problem

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Abstract

The convective-diffusive problem presents a challenge to several numerical methods. More efficient methods are available when the problem is homogeneous. Three such methods are the boundary element method (BEM), the boundary knot method (BKM), and the method of fundamental solutions (MFS). These methods require only boundary nodes or boundary mesh, but not domain nodes or domain mesh as required by the finite difference method or the finite element method. The performances of the three methods in solving a sample two-dimensional convective-diffusive problem are compared in this paper. It is found that BKM and MFS give more accurate results when the solution is a relatively smooth function, but fail when the solution varies too rapidly. On the other hand, BEM can deal with all types of solutions, and its solutions exhibit convergence.

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1. Introduction

Several physical and chemical phenomena involve both convection and diffusion. As a result, several numerical methods have been used to solve the convective-diffusive problem. The partial differential equation describing the convective-diffusive problem consists of partial derivatives in space of first order and second order, which may cause difficulty for certain mesh-dependent numerical methods such as the finite difference method and the finite element method [1]. Recently, meshless methods based on radial basis functions have been shown to provide accurate solutions to the convective-diffusive problem [2-4]. These methods use global collocation to discretize the partial differential equation, whereas mesh-dependent methods use local interpolation schemes.

For the homogeneous convective-diffusive problem, numerical methods that require only boundary nodes or boundary mesh can be used to obtain solutions. These methods are computationally more efficient than methods that require the generation of domain nodes or domain mesh. The three such methods considered in this paper are the boundary element method (BEM), the boundary knot method (BKM), and the method of fundamental solutions (MFS). BEM converts the governing partial differential equation into a boundary integral equation. Discretization of this equation using local interpolation over boundary mesh, numerical integration, and global assembling of elemental equations results in a system of linear equations to be solved for unknown variables on the boundary. The success of BEM in solving the convective-diffusive problem is well known [5]. BKM expresses the solution to the governing differential equation as a linear superposition of nonsingular fundamental solutions. Unknown coefficients in this expression can be determined by collocation at the boundary. BKM was proposed by Chen and Tanaka [6], and was used by Chen and Hon [7] to solve the convective-diffusive problems in two dimensions and three dimensions. MFS is similar to BKM, except that singular fundamental solutions are used instead, which necessitates the construction of auxiliary boundary. MFS was successfully used to solve various problems [8], but there has heretofore been no work on using MFS to solve the convective-diffusive problem. Since the three methods are similar in their requiring only boundary nodes, it is interesting to compare the perfomance of the three methods in solving this problem.

The main objective of this paper is to compare the performance of BEM, BKM, and MFS in solving a sample steady-state convective-diffusive problem in two dimensions. In the following sections, the mathematical description of the convective-diffusive problem is given, BEM, BKM, and MFS formulations for solving the problem are described, and, finally, numerical results of the three methods are compared and discussed.

2. Mathematical description of the problem

Let $u(x,y)$ be an arbitrary scalar function in a homogeneous, isotropic and in-compressible medium of domain Ω with boundary Γ having constant velocity field $\vec{v} = v_x\vec{i} + v_y\vec{j}$. If the steady state prevails, and there is no generation term, the partial differential equation describing the distribution of $u(x,y)$ in Ω is

$$\frac{\partial}{\partial x}\left(\alpha\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha\,\frac{\partial u}{\partial y}\right) - v_x\frac{\partial u}{\partial x} - v_y\frac{\partial u}{\partial y} \;=\; 0 \tag{1}$$

with boundary conditions

$$u(x,y) = f(x,y) \qquad \text{for } (x,y) \text{ on } \Gamma_1 \tag{2}$$

$$\kappa n_x\frac{\partial u}{\partial x} + \kappa n_y\frac{\partial u}{\partial y} \;=\; q(x,y) \qquad \text{for } (x,y) \text{ on } \Gamma_2 \tag{3}$$

where α is the diffusive coefficient, κ is the conductive coefficient, $n_x\vec{i} + n_y\vec{j}$ is the unit normal vector, and $f(x,y)$ and $q(x,y)$ are arbitrary functions. Equation (1) can describe several transport phenomena including heat transfer with convection, for which u is temperature, α is thermal diffusivity, and κ is thermal conductivity.

3. Numerical methods

Since Eq. (1) is homogeneous, there is no need for domain nodes or domain mesh to discretize it. A suitable method should only require boundary nodes or bound-ary mesh on Γ. This method may be BEM, BKM, or MFS.

3.1 Boundary Element Method (BEM)

Equation (1) can be transformed into the following boundary integral equation [5].

$$c\left(\vec{\bar{\xi}}\right)u\left(\vec{\bar{\xi}}\right) \;=\; \int_{\Gamma}\frac{1}{\kappa}\frac{\partial u(\vec{r})}{\partial n}\,G\left(\vec{r}-\vec{\bar{\xi}}\right)d\Gamma - \int_{\Gamma}u(\vec{r})\,\frac{\partial G\left(\vec{r}-\vec{\bar{\xi}}\right)}{\partial n}\,d\Gamma \tag{4}$$

where $\vec{\bar{\xi}}$ is the displacement vector of the location where u is evaluated, $\vec{r}$ is displacement vector of location along Γ, c is coefficient that depends on the location of $\vec{\bar{\xi}}$, and G is the fundamental solution.

$$G\left(\vec{r}-\vec{\xi}\right) \quad = \quad \frac{1}{2\pi\kappa} \exp\left(\frac{-v.\left(\vec{r}-\vec{\xi}\right)}{2\alpha}\right) K_0\left(\mu\left|\vec{r}-\vec{\xi}\right|\right) \tag{5}$$

where K_0 is the modified Bessel function of the second kind of order zero, and factor μ is

$$\mu \quad = \quad \sqrt{\frac{v_x^2 + v_y^2}{2\alpha}} \tag{6}$$

Discretizing Eq. (4) yields a set of boundary element equations, which can be solved together with boundary conditions (2) and (3) for u and $\partial u/\partial n$ at all boundary nodes.

3.2 Boundary Knot Method (BKM)

Solution to Eq. (1) can be written as a linear superposition of radial basis function ϕ.

$$u(\vec{r}) \quad = \quad \sum_{i=1}^{N} a_i\, \phi\left(\vec{r}-\vec{r}_i\right) \tag{7}$$

where N is the number of boundary nodes,

$$\phi\left(\vec{r}-\vec{r}_i\right) \quad = \quad \frac{1}{2\pi\alpha} \exp\left(\frac{\vec{v}.\left(\vec{r}-\vec{r}_i\right)}{2\alpha}\right) I_0\left(\mu\left|\vec{r}-\vec{r}_i\right|\right) \tag{8}$$

I_0 is the modified Bessel function of the first kind of order zero, and factor μ is the same as in Eq. (6) because

$$\nabla^2\phi - \vec{v}.\vec{\nabla}\phi \quad = \quad 0 \tag{9}$$

which is equivalent to Eq. (1) for two-dimensional problem in rectangular coordinates. Unknown coefficients a_i in Eq. (7) can be determined by collocation at nodes on Γ_1 and Γ_2.

$$\sum_{i=1}^{N} a_i\, \phi\left(\vec{r}_j-\vec{r}_i\right) \quad = \quad f\left(\vec{r}_j\right) \qquad \text{for } \vec{r}_j \text{ on } \Gamma_1 \tag{10}$$

$$\sum_{i=1}^{N} a_i\, \kappa\left[n_x\, \frac{\partial\phi\left(\vec{r}_j-\vec{r}_i\right)}{\partial x} + n_y\, \frac{\partial\phi\left(\vec{r}_j-\vec{r}_i\right)}{\partial y}\right] \quad = \quad q\left(\vec{r}_j\right) \ \text{ for } \vec{r}_j \text{ on } \Gamma_2 \tag{11}$$

3.3 Method of Fundamental Solutions (MFS)

Solution to Eq. (1) can also be written as a linear superposition of fundamental solutions [Eq. (5)]

$$u(\vec{r}) \quad = \quad \sum_{i=1}^{N} b_i \, G\left(\vec{r} - \vec{\rho}_i\right) \tag{12}$$

However, since

$$\nabla^2 G - \vec{v}.\vec{\nabla} G \quad = \quad \delta\left(\vec{r} - \vec{\rho}_i\right) \tag{13}$$

where δ is the Dirac-delta function, G satisfies Eq. (1) only if $\vec{\rho}_i$ is outside Ω. Therefore auxiliary boundary points are required. The number of auxiliary boundary points is equal to the number of boundary nodes N. Unknown coefficients b_i in Eq. (12) can be determined by collocation at nodes on Γ_1 and Γ_2.

$$\sum_{i=1}^{N} b_i \, G\left(\vec{r}_j - \vec{\rho}_i\right) \quad = \quad f\left(\vec{r}_j\right) \qquad \text{for } \vec{r}_j \text{ on } \Gamma_1 \tag{14}$$

$$\sum_{i=1}^{N} b_i \, \kappa \left[n_x \, \frac{\partial G\left(\vec{r}_j - \vec{\rho}_i\right)}{\partial x} + n_y \, \frac{\partial G\left(\vec{r}_j - \vec{\rho}_i\right)}{\partial y} \right] \quad = \quad q\left(\vec{r}_j\right) \quad \text{for } \vec{r}_j \text{ on } \Gamma_2 \tag{15}$$

4. Results and discussion

The sample problem to be solved by the three methods is the convective-diffusive problem in two dimensions. The domain is a square of unit length as shown in Fig. 1. Auxiliary boundary constructed for MFS is a concentric square of 1 + 2d length so that distance between boundary and auxiliary boundary is d.
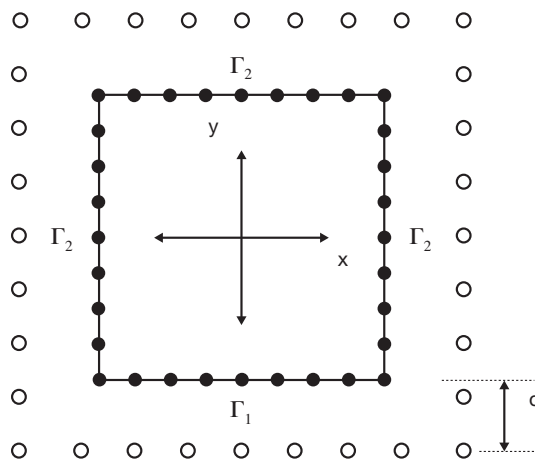

Fig. 1 The square domain of unit length for the sample two-dimensional convective-diffusive problem.

The origin of the x-y coordinates is located at the centroid of the square. Γ_1 is the bottom side of the domain, and Γ_2 is the other three sides. Boundary nodes, uniformly distributed on Γ_1 and Γ_2, are represented by solid circles, and auxiliary boundary nodes (for MFS) are represented by circles. Distance between boundary and auxiliary boundary is d. This figure shows the case in which the number of boundary nodes N is 32.

In order to determine the accuracy of the three methods, exact solution to Eq. (1) is needed. For this purpose, consider a sample problem of which governing equation is Eq. (1), and boundary conditions are

$$u(x,-0.5) \quad = \quad \exp(v_x x - 0.5 v_y) \tag{16}$$

$$\frac{\partial u(0.5,\, y)}{\partial x} \quad = \quad v_x \exp(0.5 v_x + v_y\, y) \tag{17}$$

$$\frac{\partial u(-0.5,\, y)}{\partial x} \quad = \quad v_x \exp(-0.5 v_x + v_y\, y) \tag{18}$$

$$\frac{\partial u(x, 0.5)}{\partial y} \quad = \quad v_y \exp(v_x x + 0.5 v_y) \tag{19}$$

The values of the parameters α and κ are both 1 in this sample problem. The exact solution of this problem is

$$u_{\text{exact}}(x,y) \quad = \quad \exp(v_x x + v_y\, y) \tag{20}$$

This problem is also solved numerically by the three methods for values of u_i on Γ_2. The error is the average difference between the exact and computed values of u at all boundary nodes on Γ_2, defined as follows.

$$\varepsilon \quad = \quad \left[\frac{1}{M} \sum_{i=1}^{M} \left(\frac{u_{exact}(x_i, y_i) - u_i}{u_{exact}(x_i, y_i)} \right)^2 \right]^{1/2} \tag{21}$$

where M is the number of nodes on Γ_2.

Results shown in Fig. 2 are obtained by solving the sample problem using BEM. The element used is the quadratic element, and the number of boundary nodes is varied from 8 to 128. Four sets of parameters v_x and v_y are considered. It can be seen that, for each set of v_x and v_y, error decreases

monotonically with increasing N. This is the evidence of convergence behaviors of BEM solutions. Moreover, error increases when v_x and v_y increase. This is to be expected since function $u_{exact}(x,y)$ with large values of v_x and v_y is a rapidly-varying function, which is difficult to reproduce by numerical methods. It is also well known that function $u_{exact}(x,y)$ with large values of v_x and v_y causes oscillatory behavior when the convective-diffusive problem is solved by the finite difference method [1]. Upwind technique can suppress this behavior, but it results in numerical diffusion. No such difficulties are encountered when BEM is used.

Fig. 3 shows results obtained from BKM. For small values of v_x and v_y, BKM appears to yield more accurate results than BEM. It is interesting, however, to note that error in this case does not decrease monotonically with increasing N as in the case of BEM. More importantly, numerical results show that BKM fails to produce acceptable results when v_x and v_y are equal to 20 no matter how many boundary nodes are used.
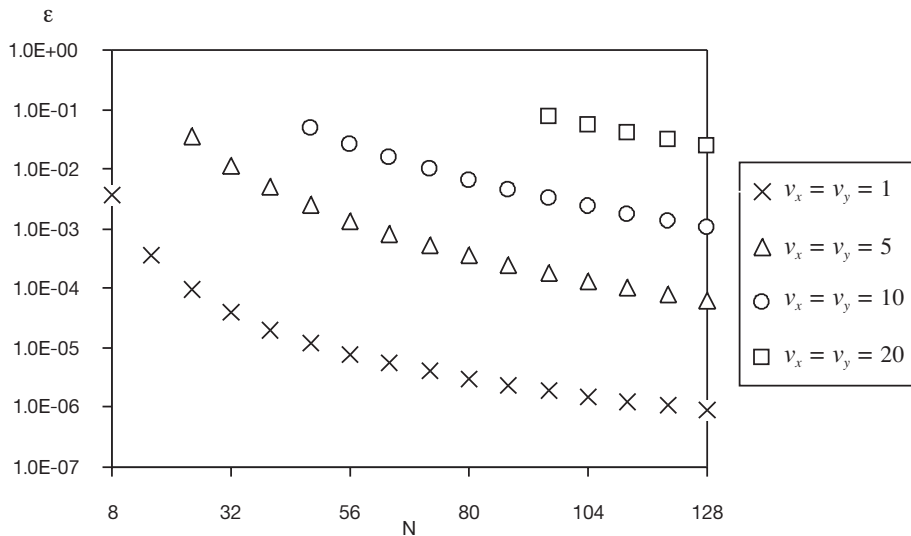

Fig. 2 Variation of error of the BEM solution with the number of boundary nodes N and parameters v_x and v_y of function $u_{exact}(x,y)$. Values of ε greater than 0.01 are not shown since they indicate the failure of the method.
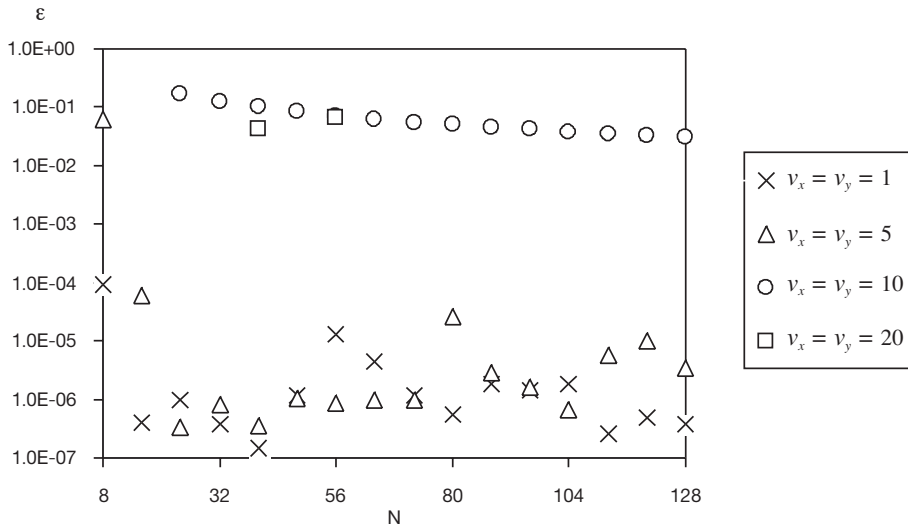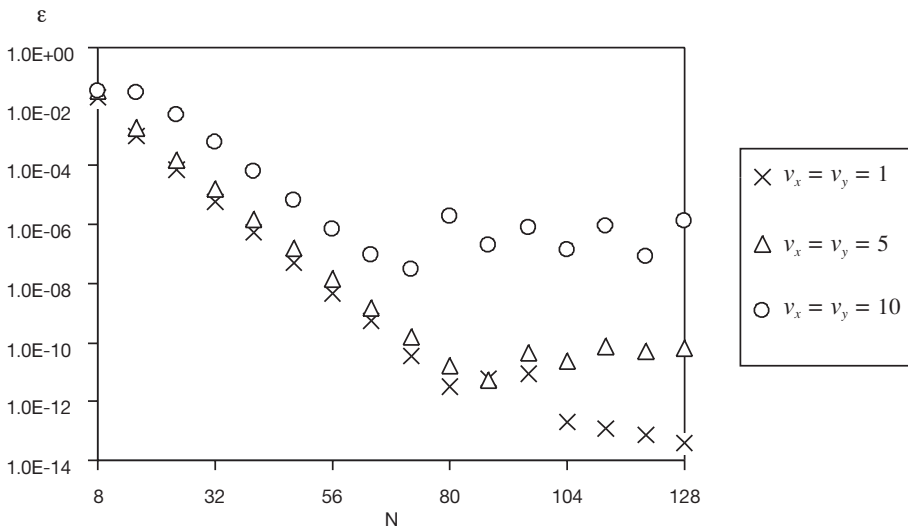
Fig. 3 Variation of error of the BKM solution with the number of boundary nodes N and parameters v_x and v_y of function $u_{\text{exact}}(x,y)$. Values of ε greater than 0.01 are not shown

Fig. 4 Variation of error of the MFS solution with the number of boundary nodes N and parameters v_x and v_y of function $u_{\text{exact}}(x,y)$. Values of ε greater than 0.01 are not shown.
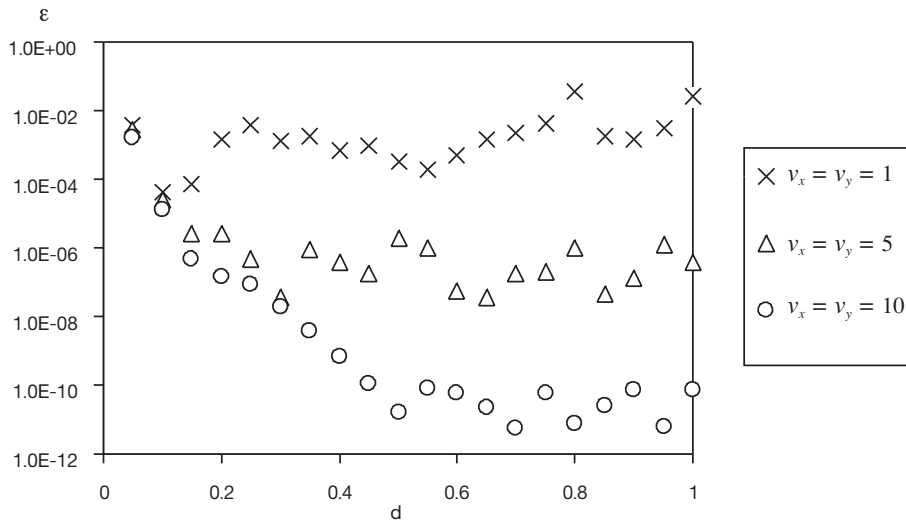
Fig. 5 Variation of error of the MFS solution with the number of boundary nodes N and the distance d between the boundary and the auxiliary boundary. Values of ε greater than 0.01 are not shown.

Finally, results for the case of MFS are shown in Fig. 4 For the sample problem, the accuracy of MFS solutions is comparable to that of BKM solutions when v_x and v_y are small. Increasing N results in smaller error until N is about 72. Further increase in N may not decrease error. Note that, like BKM, MFS also fails when v_x and v_y are equal to 20. The accuracy of MFS solutions depends not only on the N, v_x and v_y, but also on the distance d between the boundary and the auxiliary boundary. Fig. 5 shows that an increase in d produces mixed results. For $v_x = v_y = 5$, error decreases monotically until d = 0.5. For larger d and larger values of v_x and v_y, error is relatively insensitive to d.

5. Conclusions

For the problem considered, BEM with uniform mesh is superior to BKM and MFS with uniform nodal distribution. Although BEM is less accurate than BKM and MFS for relatively smooth solutions, BEM can deal with rapidly-varying solutions better than BKM and MFS. More importantly, BEM results always display convergence, whereas BKM and MFS do not.

Numerical methods can be divided into methods that yield weak-form solutions and methods that yield strong-form solutions. Since BEM is representative of the former, and BKM and MFS are representatives of the latter. The results from this investigation seem to suggest the superiority of traditional methods such as BEM and the finite element method that produce weak-form solutions over recently proposed meshless methods that produce strong-form solutions.

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6. References

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