ผลเฉลยของปัญหาปัวส์ซองไม่เชิงเส้นโดยวิธีผลเฉลยมูลฐาน

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รับเมื่อ 25 เมษายน 2548 ตอบรับเมื่อ 5 ตุลาคม 2548

บทคัดย่อ

วิธีเมชเลสสามารถใช้แก้สมการเชิงอนุพันธ์ย่อยเชิงเส้นได้อย่างมีประสิทธิภาพ งานวิจัยในปัจจุบันมุ่งศึกษาการใช้วิธีเมชเลสเพื่อแก้สมการเชิงอนุพันธ์ย่อยไม่เชิงเส้น บทความนี้นำเสนอการหาผลเฉลยของปัญหาปัวส์ซองไม่เชิงเส้นที่มีพจน์แหล่งกำเนิด ซึ่งเป็นฟังก์ชันของตัวแปรสนามและอนุพันธ์ของตัวแปรสนาม ปัญหาจะถูกแบ่งเป็นปัญหาเอกพันธ์ที่แก้ได้ด้วยวิธีผลเฉลยมูลฐาน และปัญหาไม่เอกพันธ์ที่แก้ได้ด้วยวิธีกำหนดจุดมัลติควอดริก ผลเฉลยของปัญหาได้จากวิธีการคำนวณซ้ำๆ ผลการคำนวณจากการแก้ปัญหาตัวอย่างในสามกรณีแสดงให้เห็นว่า วิธีที่นำเสนอนี้สามารถแก้ปัญหาที่มีระดับความไม่เชิงเส้นของตัวแปรสนามสูง และระดับความไม่เชิงเส้นของอนุพันธ์ย่อยของตัวแปรสนามต่ำ

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Solutions of Nonlinear Poisson Problems by the Method of Fundamental Solutions

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Received 25 April 2005 ; accepted 5 October 2005

Abstract

Meshless methods have been shown to be able to solve linear partial differential equations successfully. Recent research on meshless methods is concerned with solutions of nonlinear partial differential equations. This paper presents a method for solving the nonlinear Poisson problem with the source term being a function of both the field variable and first derivatives of the field variable. The problem is split into a homogeneous problem to be solved by the method of fundamental solutions and an inhomogeneous problem to be solved by the multiquadric collocation method. The solution is found by iteration. Results from three cases of test problems indicate that the proposed method can solve problems having a high degree of nonlinearity in the field variable, and a low degree of nonlinearity in first derivatives of the field variable.

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1. Introduction

Partial differential equations (PDE) are usually solved by numerical methods. The finite difference method (FDM) and the finite element method (FEM) are the two most popular numerical methods for solving partial differential equations. Each method has its own advantages and disadvantages. FDM is easier to understand and implement, whereas FEM is capable of handling complex geometries. Recently, alternative methods that are both easy to implement and capable of handling complex geometries have been proposed. One such method is the method of fundamental solutions (MFS). MFS has been used to solve linear homogeneous PDE [1-5]. By combining MFS with collocation methods using radial basis functions, linear inhomogeneous PDE can also be solved successfully [6-8]. An additional advantage of MFS is that it requires only specification of boundary nodes for homogeneous problems. For inhomogeneous problems, an additional requirement is the automatic generation of Cartesian grid [8]. Therefore, the preprocessing step of MFS is much simpler than that of FEM, making MFS a quite promising numerical method.

MFS must be able to handle nonlinear PDE effectively to enable it to be a strong alternative method to FDM and FEM. There have been relatively few works on using MFS to solve nonlinear problems. It has been demonstrated that MFS is capable of yielding accurate solutions to several nonlinear Poisson problems [9-11]. However, previous authors considered only nonlinear terms consisting of only the field variable. In this paper, the nonlinear Poisson problem with source term being a function that depends on the field variable and first derivatives of the field variable are considered. Such problems are solved by the proposed method, which is the combination of MFS and multiquadric collocation method, on Cartesian grid. Behaviors of solutions will be studied, which will point out limitations of the proposed method in solving nonlinear Poisson problems. In the following sections, the mathematical description of nonlinear PDE is given, and the formulation of the proposed method is described. Next, results of solutions to test problems are presented and discussed.

2. Nonlinear PDE

The nonlinear PDE to be solved over a two-dimensional domain Ω with enclosing boundary $\Gamma(=\Gamma_1 \cup \Gamma_2)$ is defined by the following equation.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = s\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) \qquad \text{for } (x,y) \text{ in } \Omega \qquad \textbf{(1)}$$

with boundary conditions

$$u(x,y) = f(x,y) \qquad \text{for } (x,y) \text{ on } \Gamma_1 \qquad \textbf{(2)}$$

$$n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} = g\,(x, y) \qquad \text{for } (x, y) \text{ on } \Gamma_2 \qquad \textbf{(3)}$$

where n_x and n_y are x- and y-components, respectively, of the outward normal unit vector. Variable u is known as the field variable. The aim of the solution to Eqs. (1) - (3) is to find the value of this field variable at a specified coordinate in the domain. It should be noted that previous uses of MFS to solve Eqs. (1) - (3) assumed that the source function in Eq. (1) is $s\,(x, y, u)$. In this study, the source function is allowed to be a function of first derivatives of the field variable $\partial u/\partial x$ and $\partial u/\partial y$ also.

The solution of Eqs. (1) - (3) consists of two parts.

$$u = u_p + u_h \qquad \textbf{(4)}$$

The original problem is split into the following problem for u_p:

$$\frac{\partial^2 u_p}{\partial x^2} + \frac{\partial^2 u_p}{\partial y^2} = s\left(x, y, u_p + u_h, \frac{\partial u_p}{\partial x} + \frac{\partial u_h}{\partial x}, \frac{\partial u_p}{\partial y} + \frac{\partial u_h}{\partial y} \right)$$

$$\text{for } (x, y) \text{ in } \Omega \text{ and on } \Gamma \qquad \textbf{(5)}$$

and the following problem for u_h:

$$\frac{\partial^2 u_h}{\partial x^2} + \frac{\partial^2 u_h}{\partial y^2} = 0 \qquad \text{for } (x, y) \text{ in } \Omega \qquad \textbf{(6)}$$

with boundary conditions

$$u_h\,(x, y) = f\,(x, y) - u_p\,(x, y) \qquad \text{for } (x, y) \text{ on } \Gamma_1 \qquad \textbf{(7)}$$

$$n_x \frac{\partial u_h}{\partial x} + n_y \frac{\partial u_h}{\partial y} = g\left(x, y\right) - n_x \frac{\partial u_p}{\partial x} - n_y \frac{\partial u_p}{\partial y}$$

$$\text{for } (x, y) \text{ on } \Gamma_2 \qquad \textbf{(8)}$$

3. Proposed method

Let nodes numbered 1, 2, …, N_{b1} be boundary nodes on Γ_1, nodes numbered N_{b1} + 1, N_{b1} + 2, …, N_b be boundary nodes on Γ_2, and nodes numbered N_b + 1, N_b + 2, …, N_b + N_i be interior nodes of domain Ω. The total number of nodes is $N = N_b + N_i$. Problems described by Eqs. (5) - (8) must be solved iteratively. Suppose that after the n^{th} iteration, values of $(u_p)_i^{(n)}$ and $(u_h)_i^{(n)}$ are known, values of $(u_p)_i^{(n+1)}$ and $(u_h)_i^{(n+1)}$ at the $(n+1)^{th}$ iteration are to be determined.

Multiquadric collocation method approximates $(u_p)_i^{(n+1)}$ as

$$(u_p)_i^{(n+1)} = \sum_{j=1}^{N} a_j^{(n+1)} \varphi\left(r_{ij}\right) \qquad (i = 1, 2, \ldots, N) \qquad \textbf{(9)}$$

where $r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ and

$$\phi(r_{ij}) = \sqrt{r_{ij}^2 + c^2} \qquad \textbf{(10)}$$

which is known as multiquadrics. Shape parameter c is a real number greater than 0. Substituting Eq. (9) into Eq. (5) results in a system of equations:

$$\sum_{j=1}^{N} a_j^{(n+1)} \left[\frac{\partial^2 \varphi(r_{ij})}{\partial x_i^2} + \frac{\partial^2 \varphi(r_{ij})}{\partial y_i^2} \right] = s_i \qquad (i = 1, 2, \ldots, N) \qquad \textbf{(11)}$$

which can be solved for $a_j^{(n+1)}$

MFS approximates $(u_h)_i^{(n+1)}$ as

$$(u_h)_i^{(n+1)} = \sum_{j=1}^{N_b} b_j^{(n+1)} G(r'_{ij}) \qquad (i = 1, 2, \ldots, N_b) \qquad \textbf{(12)}$$

where $r'_{ij} = \sqrt{(x_i - \xi_j)^2 + (y_i - \eta_j)^2}$, and (ξ_j, η_j) are coordinates of source points located outsidethe domain. If fundamental solution G is the following function:

$$G(r'_{ij}) = \ln(r'_{ij}) \qquad \textbf{(13)}$$

then Eq. (6) is satisfied automatically. Substituting Eq. (12) into Eqs. (7) and (8) results in a system of equations:

$$\sum_{j=1}^{N_b} b_j^{(n+1)} G(r'_{ij}) = f_i - u_p^{(n+1)}(x_i, y_i) \qquad (i = 1, 2, \ldots, N_{b1}) \qquad \textbf{(14)}$$

$$\sum_{j=1}^{N_b} b_j^{(n+1)} \left[n_x \frac{\partial G(r'_{ij})}{\partial x_i} + n_y \frac{\partial G(r'_{ij})}{\partial y_i} \right] = g_i - n_x \left. \frac{\partial \left(u_p^{(n+1)}\right)_i}{\partial x} \right|_{(x_i, y_i)} - n_y \left. \frac{\partial \left(u_p^{(n+1)}\right)_i}{\partial y} \right|_{(x_i, y_i)}$$

$$(i = N_{b1}+1,\ N_{b1}+2,\ \ldots,\ N_{b1}+N_{b2}) \qquad \textbf{(15)}$$

where N_{b1} and N_{b2} are the numbers of nodes on 1 and 2, respectively. Since are known, and their first derivatives can be obtained. Therefore, the right-hand sides of Eqs. (14) and (15) are known, and can be determined.

After the determination of $a_j^{(n+1)}$ and $b_j^{(n+1)}$, $(u_p)_i^{(n+1)}$ and $(u_h)_i^{(n+1)}$ can be computed. The solution at the $(n+1)^{th}$ iteration can then be compared with the solution at the n^{th} iteration. The average difference is defined as

$$\Delta = \left[\frac{1}{N} \sum_{i=1}^{N} \left(1 - \frac{u_i^{(n+1)}}{u_i^{(n)}} \right)^2 \right]^{1/2} \qquad \textbf{(16)}$$

provided that $u_i^{(n)}$ is not zero. The solution is considered to have converged when Δ is less than a small tolerance number. If the solution has not converged, values of $u_i^{(n+1)}$ to be used in the next iteration are calculated from

$$u_i^{(n+1)} = \theta u_i^{(n+1)} + (1 - \theta) u_i^{(n)} \quad \textbf{(17)}$$

where θ is the relaxation parameter. It will be seen that strongly nonlinear problems require small values of θ for convergence. This iteration scheme requires initial values of $u^{(o)}(x, y)$, which may be uniformly zero. Assume that the convergence is reached after the k^{th} iteration. The average error of the converged solution is determined from

$$\varepsilon = \left[\frac{1}{N_i} \sum_{i=N_b+1}^{N} \left(1 - \frac{u_i^{(k)}}{u_{exact,i}} \right)^2 \right]^{1/2} \quad \textbf{(18)}$$

The number k is to be referred to as the convergence iteration number.

4. Results and discussion

In examples of nonlinear PDE to be considered in this paper, functions $f(x, y)$ and $g(x, y)$ are given by

$$f(x, y) = e^{x+y} \quad \textbf{(19)}$$

$$g(x, y) = (n_x + n_y) e^{(x+y)} \quad \textbf{(20)}$$

With $f(x, y)$ and $g(x, y)$ given, the solution depends on only the source function. Three cases of source functions are considered.

Case 1: $$s = u^n + 2e^{x+y} - e^{nx+ny} \quad \textbf{(21)}$$

Case 2: $$s = \left| u \frac{\partial u}{\partial x} \right|^{n/2} + 2e^{x+y} - e^{nx+ny} \quad \textbf{(22)}$$

Case 3: $$s = \left(\frac{\partial u}{\partial x} \right)^{n} + 2e^{x+y} - e^{nx+ny} \quad \textbf{(23)}$$

The exact solution for all cases is

$$u_{exact}(x, y) = e^{x+y} \quad \textbf{(24)}$$

In problems of cases 1 - 3, the degree of nonlinearity depends on n. Since problems of cases 1 and 3 are linear if $n = 1$, this value of n is not considered here. Note that the larger n is, the more nonlinear the problem becomes. In practical problems, however, n is usually not very large. Therefore, this study considers 4 values of n from $n = 2$ to 5.

The domain of the two-dimensional test problems is the region between two concentric circles of radii 0.6 and 0.3 as shown in Fig. 1. The Cartesian grid is generated by $(0.1i, 0.1j)$, where indices i and j run from -6 to 6. N_{b1} boundary nodes are uniformly distributed on the outer circle, and N_{b2} boundary nodes are uniformly distributed on the inner circle. N_{b1} and N_{b2} are chosen to be 40 and 20, respectively, so that spacing on the boundary is comparable to Cartesian grid spacing. Furthermore, N_{b1} source points are uniformly distributed on a concentric circle (not shown) of radius

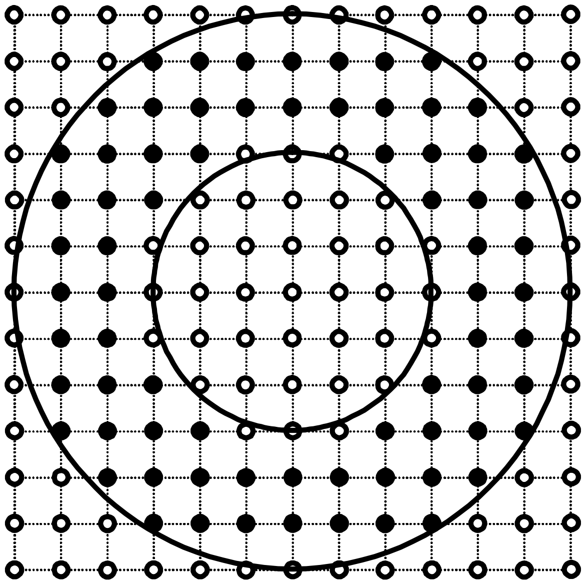

Fig. 1 Domain of two-dimensional test problems embedded in a Cartesian grid. The radii of the outer and inner circles are 0.6 and 0.3, respectively. Cartesian grid nodes selected for computation are denoted by black circles.

0.6s, and N_{b2} source points are uniformly distributed on a concentric circle (not shown) of radius 0.3/s in addition to boundary points. As can be seen from Fig. 1, Cartesian grid nodes are in the ring domain, and not too close to boundary nodes.

It was shown by Chantasiriwan [8] that solutions of Poisson, Helmholtz, and diffusion-convection equations by MFS combined with multiquadric collocation method on Cartesian grid are quite insensitive to parameter s, which depends on locations of source points, and the shape parameter c of multiquadrics. Results from the present study indicate that solutions to nonlinear problems by the proposed method have similar behaviors. Hence, s and c are chosen arbitrarily to be 1.5 and 0.5, respectively.

Fig. 2 shows variations of convergence iteration number with relaxation parameter for problems of case 1. The tolerance number is 1.0×10^{-4}. Solution converges quickly when n = 2, 3 or 4. When n = 5, however, solution is more difficult to converge. Parameter θ clearly influences convergence behaviors of solutions. When n = 2, 3 or 4, this parameter may be chosen to be 1.0. But when n = 5, the value of θ = 1.0 is not a good choice. An optimization scheme may be needed to find the value of θ that yields the fastest convergence. Consideration of such a scheme is beyond the scope of the present study.

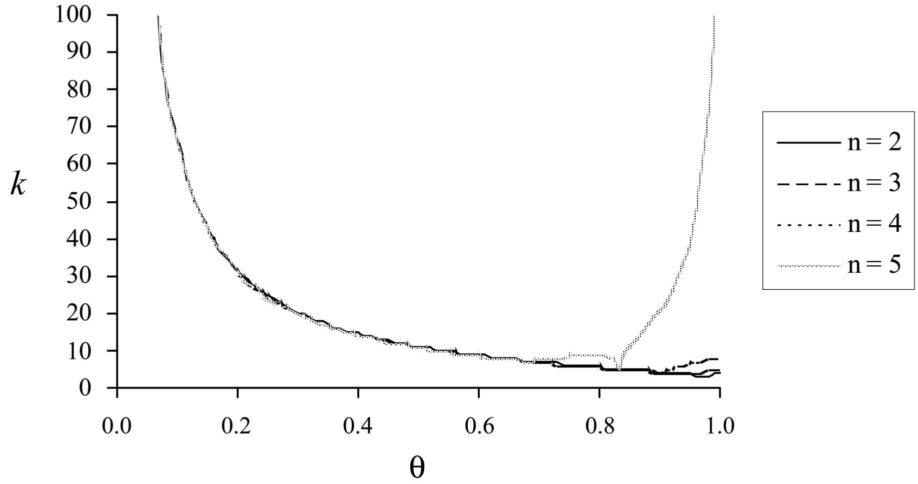

Fig. 2 Variations of k with θ for problems of case 1.

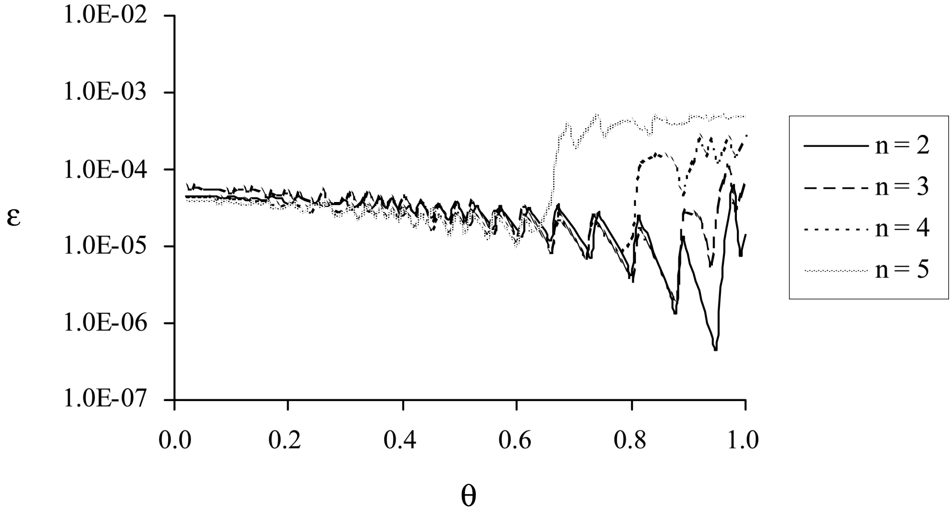

Fig. 3 Variations of ε with θ for problems of case 1.

Fig. 3 shows variations of solution error with relaxation parameter for problems of case 1. It can be seen that the solution of a problem with lower value of n is more accurate than the solution of a problem with higher value of n. In other words, the solution becomes less accurate as the problem is more nonlinear. Nevertheless, accuracies of all solutions to problems of case 1 by the proposed method are satisfactory. It can then be concluded that the proposed method can solve a nonlinear Poisson problem in which the source term is a function of the field variable. Previous attempts to solve similar nonlinear Poisson problems by MFS yielded similar results [9-11].

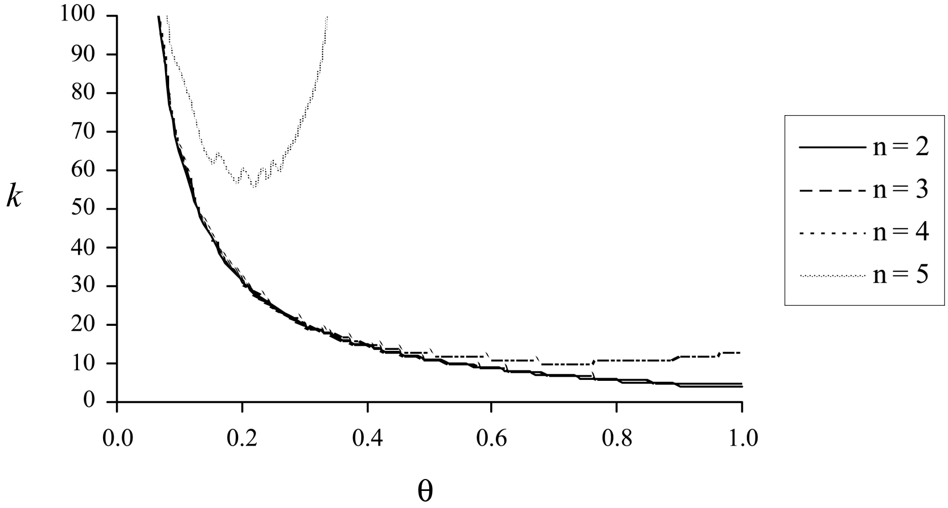

Fig. 4 Variations of k with θ for problems of case 2.

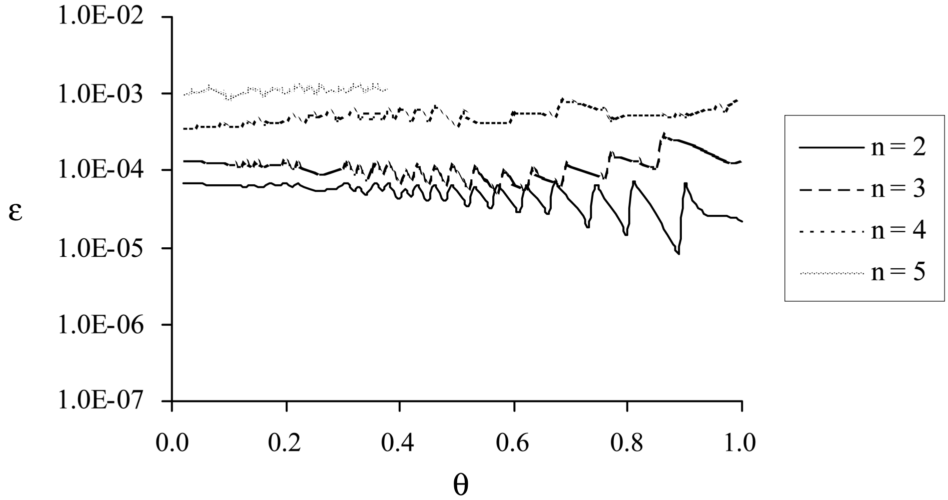

Fig. 5 Variations of ε with θ for problems of case 2.

Fig. 4 shows variations of convergence iteration number with relaxation parameter for problems of case 2. When n = 2 or 3, solution converges quickly. Convergence rate is slower when n = 4. However, convergence rate is too slow when n = 5. In fact, there is no convergence for θ > 0.38. It can therefore be said that the proposed method fails to solve the problem of case 2 with n = 5 satisfactorily. In general, the proposed method has more difficulty with problems of case 2 than problems of case 1. This difficulty is probably due to the first derivative of the field variable in the source function.

Fig. 5 shows variations of solution error with relaxation parameter for problems of case 2. Compared with solutions to problems of case 1, solutions to problems of case 2 are less accurate. However, behaviors of solutions to problems of case 1 and case 2 are similar in that the solution becomes less accurate as the problem is more nonlinear. Accuracies of all solutions to problems of case 2 for n = 2, 3 or 4 by the proposed method are satisfactory. It can then be concluded that the proposed method can solve a nonlinear Poisson problem in which the source term is a function of the product of the field variable and the first derivative of the field variable provided that the degree of nonlinearity is not too large.

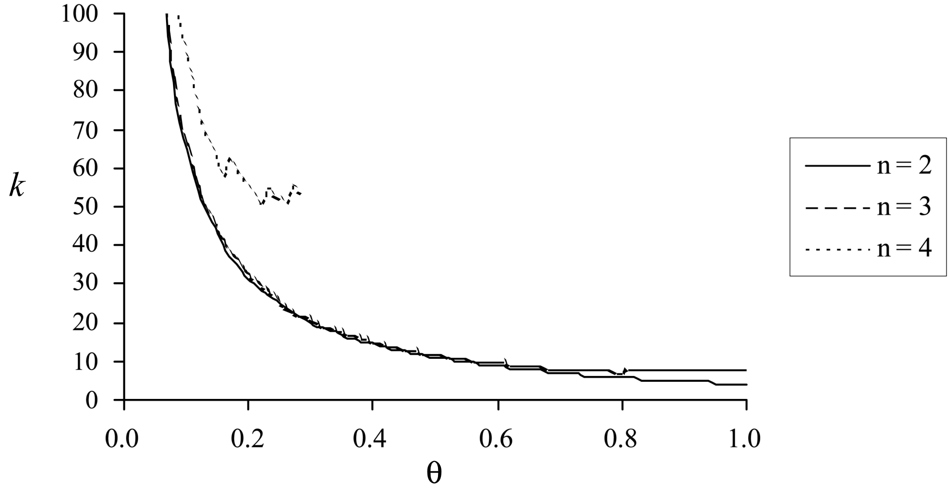

Fig. 6 Variations of k with θ for problems of case 3.

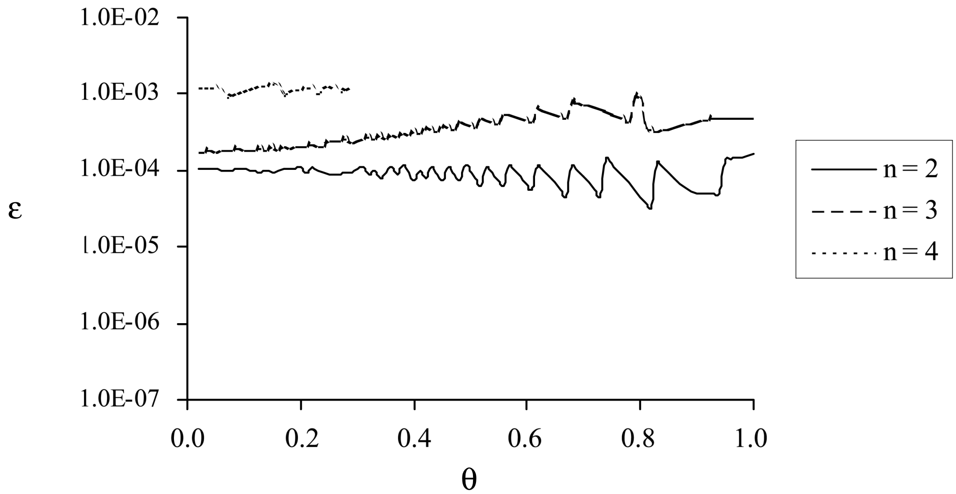

Fig. 7 Variations of ε with θ for problems of case 3.

Results for problems of case 3 are shown in Figs. 6 and 7. Results for n = 5 are not shown in these figures because no converged solution exists. For n = 4, there is no convergence when $\theta > 0.29$. The proposed method appears to have the most difficulty with problems of case 3. This is probably due to the fact that the degree of nonlinearity in the first derivative of the field variable of in source term of problems of case 3 is even stronger than that of problems of case 2. It can be concluded from results in Figs. 4 - 7 that the proposed method can solve a nonlinear Poisson problem provided that the degree of nonlinearity in first derivative of the field variable is not too large.

5. Conclusions

In this paper, a method combining the method of fundamental solutions and the multiquadric collocation method is presented and used to solve the nonlinear Poisson problem with the source function depending on the field variable and first derivatives of the field variable. It is found that, as the problem becomes more nonlinear, the solution becomes less accurate, and the number of iterations required for convergence increases. Moreover, results indicate that the proposed method has more difficulty with nonlinearity in first derivatives of the field variable than nonlinearity in the field variable. Future investigation should focus on improving the capability of this method to deal with nonlinearity in the source function of the nonlinear Poisson equation.

6. Acknowledgement

This work is financially supported by the financial support from the Thailand Research Fund.

7. References

1. Johnson, R. L. and Fairweather, G. 1984, "The method of Fundamental Solutions for Problems in Potential Flow," *Applied Mathematical Modelling*, Vol. 8, pp. 265-270.

2. Karageorghis, A. and Fairweather, G., 1987, "The Method of Fundamental Solutions for the Numerical Solution of the Biharmonic Equation," *Journal of Computational Physics,* Vol. 69, pp. 434-459.

3. Golberg, M. A., 1995, "The Method of Fundamental Solutions for Poisson's Equation," *Engineering Analysis with Boundary Elements*, Vol. 16, pp. 205-213.

4. Balakrishnan, K. and Ramachandran, P. A., 2000, "The Method of Fundamental Solutions for Linear Diffusion-reaction Equations," *Mathematical and Computer Modelling,* Vol. 31, pp. 221-237.

5. Poullikkas, A., Karageorghis, A., and Georgiou, G., 2002, "The Method of Fundamental Solutions for Three-dimensional Elastostatics Problems," *Computers and Structures,* Vol. 80, pp. 365-370.

6. Golberg, M. A., Chen, C. S., and Karur, S. R., 1996, “Improved Multiquadric Approximation for Partial Differential Equations,” *Engineering Analysis with Boundary Elements*, Vol. 18, pp. 9-17.

7. Li, J., Hon, Y. C., and Chen, C. S., 2002, “Numerical Comparisons of Two Meshless Methods using Radial Basis Functions,” *Engineering Analysis with Boundary Elements*, Vol. 26, pp. 205-225.

8. Chantasiriwan, S., 2004, “Cartesian Grid Methods using Radial Basis Functions for Solving Poisson, Helmholtz, and Diffusion-Convection Equations,” *Engineering Analysis with Boundary Elements,* Vol. 28, pp. 1417-1425.

9. Chen, C. S., 1995, “The Method of Fundamental Solutions for Non-linear Thermal Explosions,” *Communications in Numerical Methods in Engineering*, Vol. 11, pp. 675-681.

10. Balakrishnan, K. and Ramachandran, P. A., 1999, “A Particular Solution Trefftz Method for Non-linear Poisson Problems in Heat and Mass Transfer,” *Journal of Computational Physics,* Vol. 150, pp. 239-267.

11. Balakrishnan, K. and Ramachandran, P. A., 2001, “Osculatory Interpolation in the Method of Fundamental Solution for Nonlinear Poisson Problems,” *Journal of Computational Physics,* Vol. 172, pp. 1-18.