ระเบียบวิธีไฟไนต์เอลิเมนต์แบบตัวแปรไร้จุดต่อ สำหรับปัญหาการถ่ายเทความร้อนใน 2 มิติ

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บทคัดย่อ

บทความนี้กล่าวถึงระเบียบวิธีไฟไนต์เอลิเมนต์แบบตัวแปรไร้จุดต่อทำนายการกระจายอุณหภูมิของปัญหาการถ่ายเทความร้อนใน 2 มิติ เริ่มจากการอธิบายทฤษฎีการถ่ายเทความร้อนใน 2 มิติ การหาสมการไฟไนต์เอลิเมนต์แบบตัวแปรไร้จุดต่อ ขั้นตอนการคำนวณและเงื่อนไขขอบเขต จากนั้นทำการตรวจสอบความถูกต้องของโปรแกรมที่ประดิษฐ์ขึ้นกับปัญหาแผ่นสี่เหลี่ยมผืนผ้าที่กำหนดอุณหภูมิแบบพีริออดิก ปัญหาแผ่นสี่เหลี่ยมผืนผ้าที่กำหนดอุณหภูมิแบบพีริออดิกและมีความร้อนเกิดขึ้นภายในแผ่น และปัญหาการเกิดความร้อนขึ้นในลูกตาขณะผ่าตัดสลายต้อกระจก ผลลัพธ์แสดงให้เห็นถึงประสิทธิภาพของระเบียบวิธีไฟไนต์เอลิเมนต์แบบตัวแปรไร้จุดต่อที่ใช้ทำนายการกระจายอุณหภูมิ

คำสำคัญ : ระเบียบวิธีไฟไนต์เอลิเมนต์แบบตัวแปรไร้จุดต่อ / การถ่ายเทความร้อนใน 2 มิติ

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Nodeless Variables Finite Element Method for 2D Heat Transfer Problems

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Abstract

Nodeless variables finite element method is presented to predict the temperature distribution for heat transfer problems. The paper first describes 2D heat transfer theory. The finite element formulations based on nodeless variables, the computational procedure and its boundary conditions are then represented. The validated examples with analytical solution of the proposed technique are a rectangular plate with periodic temperature problem, a rectangular plate with internal heat generation problem as well as a periodic temperature problem and heat generation in eye during phacoemulsification. The solutions show that the nodeless variables finite element method can be employed to predict the temperature distribution efficiently.

Keywords : Nodeless Variables Finite Element Method / 2D Heat Transfer

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1. Introduction

Finite element method is applied to solve heat transfer problems for a decade. Steady–state heat transfer [1] can be predicted by applying the method of weighted residual (MWR). The various element types and their element interpolation functions are widely developed such as 3–node triangle with linear element interpolation function, 6–node triangle with quadratic element interpolation function, 10–node triangle with cubic element interpolation function, and etc. [2]

This paper presents nodeless variables finite element method to solve 2D steady state heat transfer problem. Triangular elements with nodeless variables and their element interpolation functions are described. Then, the computational procedure and its boundary conditions are shown. Next, the computational solutions by nodeless variables finite element method are validated with the exact solution and the finite element solutions using linear triangular element, respectively. The performance of the proposed method is shown in the problem of heat generation in eye during phacoemulsification.

2. Theory

2.1 Governing differential equation [1]

Steady–state heat transfer problem is governed by energy equation as following.

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=Q \qquad \textbf{(1)}$$

where k is the thermal conductivity, T is the temperature and Q is the internal heat generation.

2.2 Element interpolation functions and finite element matrices [2 – 4]

Triangular element consists of 3 nodes and 3 nodeless variables per element as shown in Fig. 1.

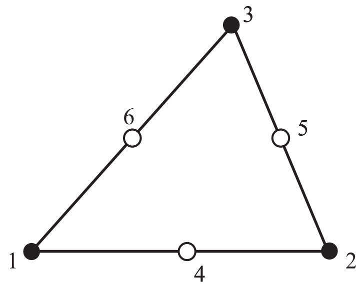

Fig. 1 Element with nodeless variables and its connectivity.

Its element interpolation functions are in the form,

$$N_i(x,y) = \frac{1}{2A}(a_i + b_i x + c_i y) \quad i = 1, 3 \qquad \textbf{(2)}$$

$$N_4 = 4N_1 N_2 \;\; ; \;\; N_5 = 4N_2 N_3 \;\; ; \;\; N_6 = 4N_1 N_3 \qquad \textbf{(3)}$$

where

$$a_i = x_j y_k - x_k y_j \;,\; b_i = y_j - y_k \;,$$

$$c_i = x_k - x_j \qquad i, j = 1,3$$

a_i, b_i, and c_i coefficients are obtained by cyclically permuting the subscripts, and A is the triangular area.

After applying MWR in equation 1, the finite element equation based on Bubnov–Galerkin method is obtained to solve 2D heat transfer problem.

$$([K_c]+[K_h])\{T\} = \{Q_c\}+\{Q_Q\}+\{Q_q\}+\{Q_h\} \qquad \textbf{(4)}$$

where

$$[K_c] = \int_A [B]^T[k][B]\,dA$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} \end{bmatrix}$$

$$\frac{\partial N_1}{\partial x} = \frac{y_2 - y_3}{2A}, \quad \frac{\partial N_1}{\partial y} = \frac{x_3 - x_2}{2A}$$

$$\frac{\partial N_2}{\partial x} = \frac{y_2 - y_1}{2A}, \quad \frac{\partial N_2}{\partial y} = \frac{x_1 - x_3}{2A}$$

$$\frac{\partial N_3}{\partial x} = \frac{y_1 - y_2}{2A}, \quad \frac{\partial N_3}{\partial y} = \frac{x_2 - x_1}{2A}$$

$$\frac{\partial N_4}{\partial x} = \frac{(y_2 - y_3)N_2 + 2(y_3 - y_1)N_1}{A}$$

$$\frac{\partial N_4}{\partial y} = \frac{(x_3 - x_2)N_2 + 2(x_1 - x_3)N_1}{A}$$

$$\frac{\partial N_5}{\partial x} = \frac{(y_3 - y_1)N_3 + 2(y_1 - y_2)N_2}{A}$$

$$\frac{\partial N_5}{\partial y} = \frac{(x_1 - x_3)N_3 + 2(x_2 - x_1)N_2}{A}$$

$$\frac{\partial N_6}{\partial x} = \frac{(y_1 - y_2)N_1 + 2(y_2 - y_3)N_3}{A}$$

$$\frac{\partial N_6}{\partial y} = \frac{(x_2 - x_1)N_1 + 2(x_3 - x_2)N_3}{A}$$

$$[K_h] = \int_S h\{N\}[N]\,dA$$

$$\{Q_c\} = \int_S \left(k\frac{\partial T}{\partial x}n_x + k\frac{\partial T}{\partial y}n_y\right)\{N\}\,dS$$

$$\{Q_Q\} = \int_A Q\{N\}\,dA$$

$$\{Q_q\} = \int_S q_s\{N\}\,dS$$

$$\{Q_h\} = \int_S hT_\infty\{N\}\,dS$$

3. Applications

A rectangular plate with sinusoidal thermal load is presented as the first example to validate nodeless variables finite element method for 2D steady–state heat transfer. The results gained from the nodeless variables finite element calculation are compared to the exact solution and the approximation solutions using 3–node triangular element. Then, applying internal heat generation in plate to evaluate the performance of nodeless variables finite element to solve 2D steady–state heat transfer problem. The third problem, heat generation in eye during phacoemulsification, is solved to express the performance of the proposed method.

3.1 Rectangular plate with sinusoidal temperature

Rectangular plate with dimension 0.5x1 unit is applied with a sinusoidal temperature function at the top edge and the constant zero temperature is applied along the left edge as shown in Fig. 2.

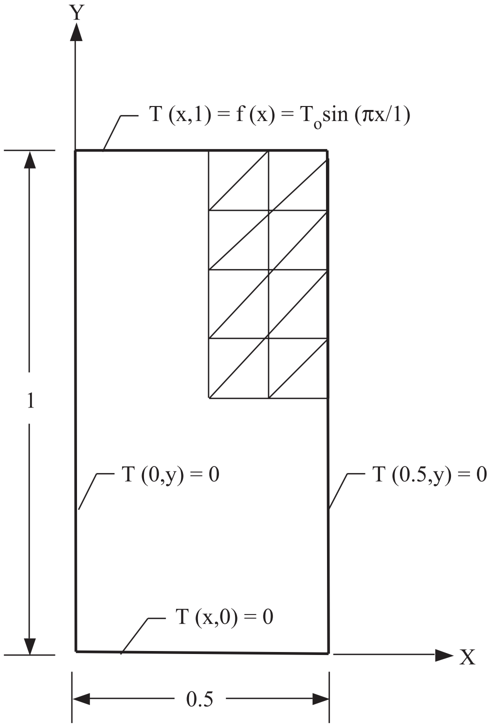

Fig. 2 Problem statement of rectangular plate.

This problem has an exact solution in the form,

$$T(x,y) = \frac{\sin(2\pi x) \times \sinh(2\pi y)}{\sinh(2\pi)} \quad \textbf{(5)}$$

where $T_o = 1.0$

After applying nodeless variables finite element method, the solution of temperature distribution is shown in Fig. 3.

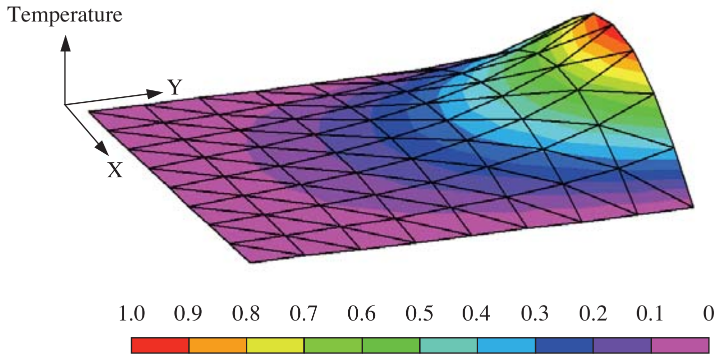

Fig. 3 Temperature distribution on rectangular plate.

To validate the accuracy of nodeless variables finite element method, the rectangular plate is discretized in several finite element models i.e., 32 elements, 48 elements, 64 elements, 72 elements, 96 elements, 112 elements, 256 elements, and 400 elements as shown in Fig. 4.

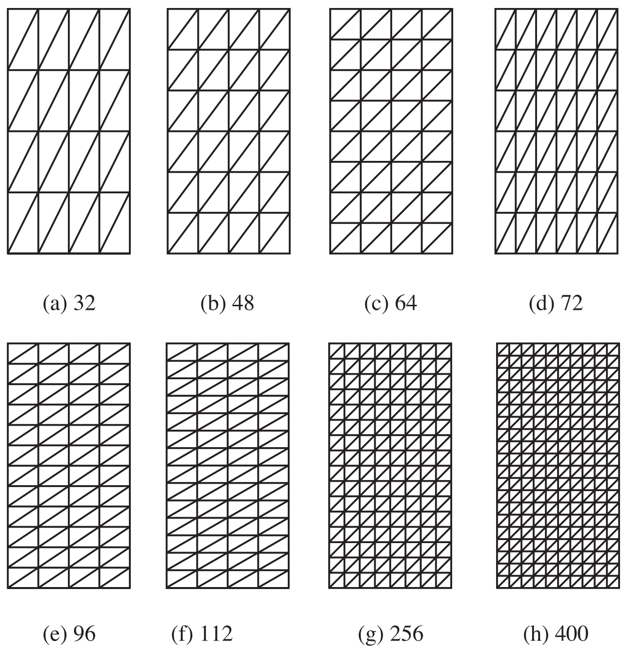

Fig. 4 Finite element number of rectangular plate.

Fig. 5 shows temperature solution along x direction at y = 0.5 of nodeless variables finite element solution comparing with exact solution and linear triangular element solution. Fig. 6 displays temperature solution along y direction at x = 0.25. The results express a good accuracy of nodeless variables finite element method in both of 32 elements and 400 elements.

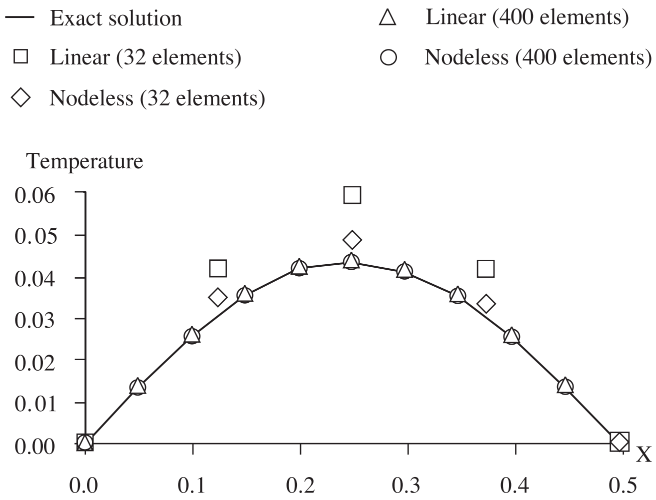

Fig. 5 Temperature distribution along x axis at y = 0.5.

The computational error has been collected in all models and displayed in Fig. 7 and 8. The results show that element with nodeless variables has fewer errors than linear element in all element number.

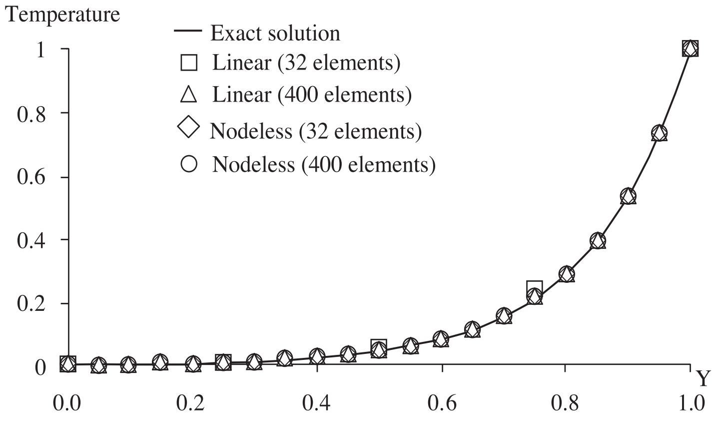

Fig. 6 Temperature distribution along y axis at x = 0.25

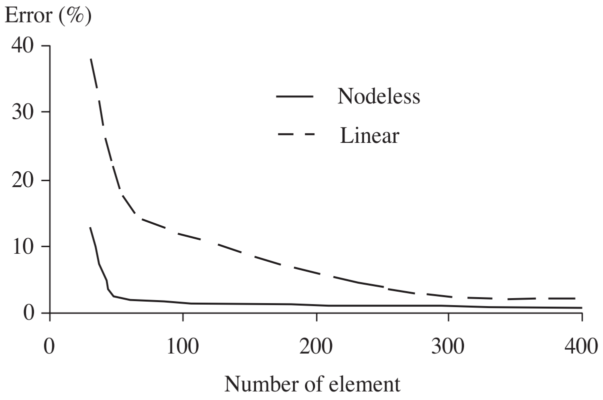

Fig. 7 Error of temperature along x axis at y = 0.5

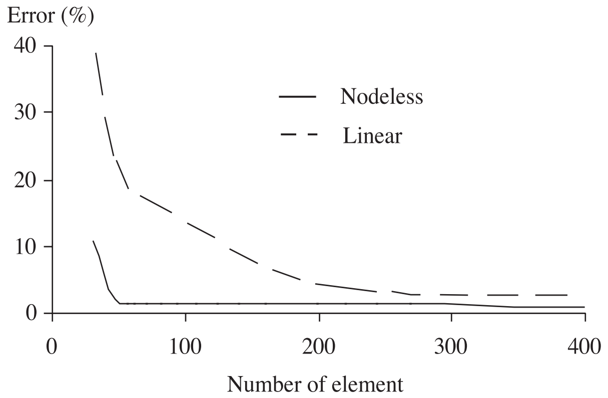

Fig. 8 Error of temperature along y axis at x = 0.25

3.2 Rectangular plate with sinusoidal temperature and internal heat generation

This example is more complicated than the previous example. The dimension of geometry is similar to the first example. Thermal load consists of sinusoidal temperature, same as the first example, at the top edge and the internal heat generation.

The exact solution of this problem is in the infinite series form as shown below,

$$T(x,y) = \frac{(1-(x-0.5a)^2)}{2} - \frac{16}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos((2n+1)\pi(x-0.5a)/2)\cos((2n+1)\pi(y-0.5b)/2)}{(2n+1)^3\cos((2n+1)\pi/2)} + \frac{\sin(2\pi x)\times\sinh(2\pi y)}{\sinh(2\pi)} \qquad \textbf{(6)}$$

where a = 1.0, b = 1.0

Finite element model consists of 400 elements and 231 nodes. Fig. 9 shows temperature solution computed by nodeless variables finite element method.

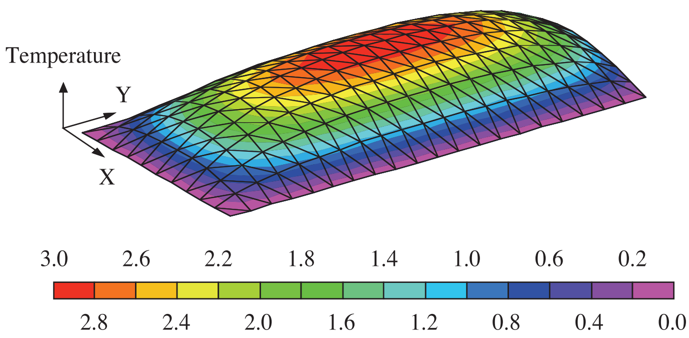

Fig. 9 Temperature distribution of second problem.

The computational solution is plotted with respect to distance in x direction at y = 0.5 and y direction at x = 0.25 as shown in Fig. 10 and 11, respectively.

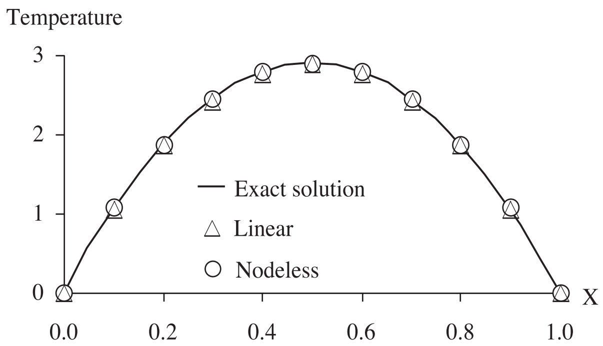

Fig. 10 Temperature distribution along x axis at y = 0.5 of rectangular plate with sinusoidal temperature at top edge and internal heat generation.

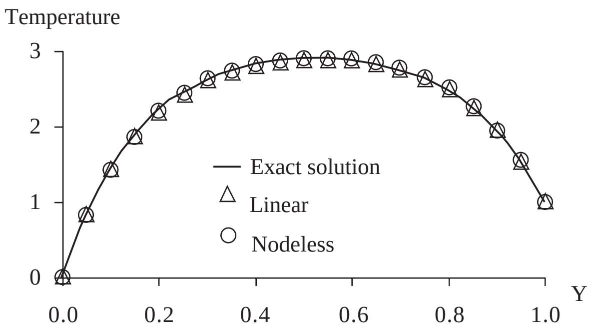

Fig. 11 Temperature distribution along y axis at x = 0.25 of rectangular plate with sinusoidal temperature at top edge and internal heat generation.

3.3 Heat generation in eye during phacoemulsification [5]

During ophthalmic phacoemulsification operation heat generation occurs at phaco needle. As ophthalmologist applies power through phaco needle. Heat generation from phaco needle's ultrasonic vibration is applied to be heat flux, 1.675 kW/m^2 , of eye domain. Anterior chamber is fully filled with balanced salt solution. At the outside surface of cornea is exposed to air condition room, T_∞ = 25 °C, and has convection heat transfer. Two dimension of globe model was drawn follow ocular anatomy [6]. Specific heat of balanced salt solution is measured by differential scanning calorimeter and thermal conductivity by transient plane source method. The problem statement of this application is shown in Fig. 12. Finite element model is then constructed with 9,204 nodes and 9,311 elements as depicted in Fig. 13. Temperature solution is then represented in Fig. 14.

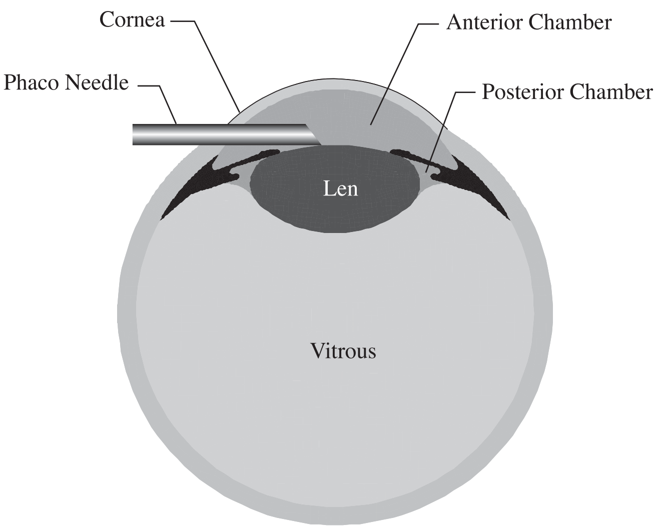

Fig. 12 Eye composition

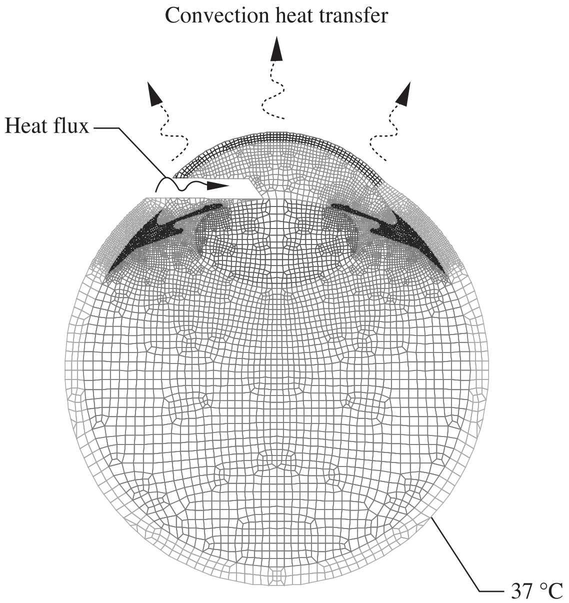

Fig. 13 Finite element model of eye

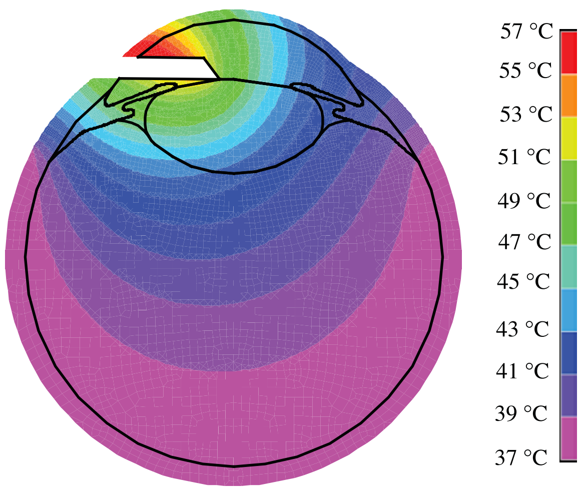

Fig. 14 Temperature distributions in eye

4. Conclusions

Nodeless variables finite element method is presented to predict 2D heat transfer problem. The proposed method is validated with three examples and results from both exact solution and the results from finite element method using linear triangular element are compared. The results clearly show that the nodeless variables element method give a higher accuracy than the linear element in solving 2D steady-state heat transfer problem.

5. Acknowledgement

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6. References

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