ผลเฉลยของปัญหาการไหลที่ขับเคลื่อนด้วยแรงลอยตัว โดยใช้วิธีการจัดตำแหน่งจุดเฉพาะที่ซึ่งมีฟังก์ชันมัลติควอดริกเป็นฟังก์ชันฐานเชิงรัศมี

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บทคัดย่อ

วิธีการจัดตำแหน่งจุดเฉพาะที่ซึ่งมีฟังก์ชันมัลติควอดริกเป็นฟังก์ชันฐานเชิงรัศมีเป็นวิธีเชิงตัวเลขที่ไม่ใช้เมช บทความนี้นำเสนอการใช้วิธีนี้ในการแก้ปัญหาการไหลที่ขับเคลื่อนด้วยแรงลอยตัวภายในช่องว่างรูปสี่เหลี่ยมจัตุรัสและภายในช่องว่างรูปวงแหวน รูปแบบของปัญหาเป็นรูปแบบที่มีตัวแปรตามเพียงสองตัวคือ ตัวแปรสายธารและตัวแปรกระแสวน อย่างไรก็ตามรูปแบบนี้ไม่ได้กำหนดเงื่อนไขขอบเขตของตัวแปรกระแสวนมาให้ ดังนั้นเงื่อนไขขอบเขตของตัวแปรกระแสวนต้องคำนวณจากนิยามเชิงคณิตศาสตร์ของกระแสวน บทความนี้จึงได้นำเสนอวิธีคำนวณค่าตัวแปรกระแสวนที่ขอบเขตซึ่งเหมาะกับวิธีการจัดตำแหน่งจุดเฉพาะที่ ผลการแก้ปัญหาในช่องว่างรูปสี่เหลี่ยมจัตุรัสแสดงให้เห็นว่าวิธีการจัดตำแหน่งจุดเฉพาะที่สามารถให้ผลเฉลยที่แม่นยำ นอกจากนี้ผลการแก้ปัญหาในช่องว่างรูปวงแหวนในกรณีที่อัตราส่วนระหว่างเส้นผ่าศูนย์กลางนอกและเส้นผ่าศูนย์กลางในเท่ากับ 1.5 และ 2, Pr = 0.7 และ Ra มีค่าระหว่าง 10^5 กับ 10^6 แสดงให้เห็นว่าวิธีการจัดตำแหน่งจุดเฉพาะที่ให้ผลเฉลยที่ใกล้เคียงกับผลการทดลอง

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Solutions of Buoyancy-driven Flow Problems by the Local Collocation Method That Uses Multiquadrics as the Radial Basis Function

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Abstract

The local multiquadric collocation method is a meshless method that uses radial basis functions known as multiquadrics to approximate functions and their derivatives. In this paper, buoyancy-driven flow problems in a square cavity and a horizontal concentric annulus are solved by this method. The stream function-vorticity formulation is used because there are only two unknowns in this formulation. However, since the vorticity boundary condition is required, but not explicitly given, it must be determined by using the definition of vorticity. A scheme for computing boundary vorticity that is appropriate to the local multiquadric collocation method is presented. Results from the buoyancy-driven flow problem in a square cavity show that the accuracy of solutions obtained by using this scheme is comparable with the accuracy of solutions obtained by using a more accurate scheme. Furthermore, it is also shown that numerical results of the buoyancy-driven flow problem in a horizontal concentric annulus for cases of D_o/D_i = 1.5 and 2.0, Pr = 0.7, and Ra_{Di} between 10^5 and 10^6 by the local multiquadric collocation method agree with experimental results

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1. Introduction

The point interpolation method introduced by Liu [1] uses a few basis functions to approximate a function and its derivatives in terms of functional values at nearby nodes. Since this method requires matrix inversion, basis functions must be carefully chosen to avoid singular matrix. Polynomial basis functions are a convenient choice, but experience has shown that they may yield a singular matrix when nodes are orderly arranged. An algorithm such as the matrix triangularization method [2] is required in this situation so that the modified matrix is invertible. On the other hand, radial basis functions yield non-singular matrix for virtually any node arrangement. Radial basis functions are therefore a better choice as far as the robustness of the method is concerned. Multiquadrics is a well-known radial basis function that was initially introduced for multivariate data interpolation. It has recently been used as basis functions for collocation methods. Local multiquadric collocation method is the point interpolation method that uses multiqudrics as basis function. This method has been successfully used to solve certain linear and nonlinear problems [3 - 5].

Buoyancy-driven flow problems are nonlinear problems that have been tested with several numerical methods. These problems describe natural convection phenomena in a variety of geometries. The simplest problem is the buoyancy-driven flow problem in a square cavity. Although this problem is simple to describe and formulate, it does not have the analytical solution. Its simple geometry allows the use of the finite difference method, which can be shown to solve the problem efficiently. When the problem domain is irregular, however, the implementation of the finite difference method may be awkward. By contrast, the local multiquadric collocation method can handle an irregular geometry more efficiently.

This paper is concerned with solutions to buoyancy-driven flow problems in the stream function-vorticity formulation by the local multiquadric collocation method. Two-dimensional buoyancy-driven flow problems may be solved in three formulations: the primitive-variable formulation, the velocity-vorticity formulation, and the stream function-vorticity formulation. The stream function-vorticity formulation is arguably the most efficient formulation of the two-dimensional Navier-Stokes equations because it reduces the three continuity and momentum equations of pressure and two velocity components into two equations of stream function and vorticity, whereas the other two formulations yield three equations of three unknowns. The following sections present the mathematical formulation of buoyancy-driven flow problems, the implementation of the local collocation method to solve these problems, and numerical results that demonstrate the effectiveness of using this method to solve natural convection problems in a square cavity and a horizontal concentric annulus.

2. Governing Equations

Two-dimensional buoyancy-driven flow problems are governed by the following continuity, momentum, and energy equations:

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \quad \textbf{(1)}$$

$$\rho\left(\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right) = -\frac{\partial p}{\partial x'}+\mu\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}\right) \quad (2)$$

$$\rho\left(\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}\right) = -\frac{\partial p}{\partial y'}+\mu\left(\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}\right)-\rho g \quad (3)$$

$$\frac{\partial T'}{\partial t'}+u'\frac{\partial T'}{\partial x'}+v'\frac{\partial T'}{\partial y'} = \alpha\left(\frac{\partial^2 T'}{\partial x'^2}+\frac{\partial^2 T'}{\partial y'^2}\right) \quad (4)$$

where ρ is density, μ is dynamic viscosity, α is thermal diffusivity, and g is gravitational acceleration. Initially, the fluid temperature is uniformly T_0. At time $t' > 0$, part of boundary is subjected to isothermal boundary condition T_w, part of boundary is kept at the initial temperature, and the other part of boundary is insulated.

In Boussinesq approximation, ρ is assumed to be constant in Eqs. (2) and (3) except in the source term of Eq. (3), where ρ is approximated as

$$\rho = \rho_0-\rho\beta(T-T_0) \quad (5)$$

Thermal expansion coefficient is defined as

$$\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p \quad (6)$$

Insert ρ from Eq. (5) into the source term of Eq. (3), and combine the resulting equation with Eqs. (1) and (2) into two equations of stream function (ψ') an vorticity (ω'):

$$\frac{\partial^2\psi'}{\partial x'^2}+\frac{\partial^2\psi'}{\partial y'^2} = -\omega' \quad (7)$$

$$\rho\left(\frac{\partial\omega'}{\partial t'}+\frac{\partial\psi'}{\partial y'}\frac{\partial\omega'}{\partial x'}-\frac{\partial\psi'}{\partial x'}\frac{\partial\omega'}{\partial y'}\right) = \mu\left(\frac{\partial^2\omega'}{\partial x'^2}+\frac{\partial^2\omega'}{\partial y'^2}\right)+\rho g\beta\frac{\partial(T'+T_0)}{\partial x'} \quad (8)$$

Equation (4) may be rewritten as

$$\frac{\partial T'}{\partial t'}+\frac{\partial\psi'}{\partial y'}\frac{\partial T'}{\partial x'}-\frac{\partial\psi'}{\partial x'}\frac{\partial T'}{\partial y'} = \alpha\left(\frac{\partial^2 T'}{\partial x'^2}+\frac{\partial^2 T'}{\partial y'^2}\right) \quad (9)$$

where

$$\omega' = \frac{\partial v'}{\partial x'}-\frac{\partial u'}{\partial y'} \quad (10)$$

$$u' = \frac{\partial\psi'}{\partial y'} \quad (11)$$

$$v' = -\frac{\partial\psi'}{\partial x'} \quad (12)$$

Assume that L is the characteristic length scale for the problem. Let's define the following dimensionless variables: $x = x'/L$, $y = y'/L$, $t = \alpha t'/L^2$, $u = u'L/\alpha$, $v = v'L/\alpha$, $T = (T'-T_0)/(T_w - T_0)$, $\omega = \alpha\omega'/L^2$, and $\psi = \psi'/\alpha$ Eqs. (7) - (9) may be written in the following dimensionless forms.

$$\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2} = -\omega \tag{13}$$

$$\frac{\partial\omega}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} = \Pr\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) + \mathrm{Ra}\,\Pr\frac{\partial T}{\partial x} \tag{14}$$

$$\frac{\partial T}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \tag{15}$$

where Rayleigh number is $\mathrm{Ra} = g\beta|T_w - T_0|L^3/\alpha^2$, and Prandtl number is $\Pr = v/\alpha$.

3. Local Multiquadric Collocation Method

Let node 1 be where the partial derivative of a function f is to be discretized. Consider a group of N interpolation nodes, which include node 1 and other N - 1 nodes that may be selected by their proximity to node 1 or by another criterion. A given functional value at each node may be approximated by

$$f(x_i,y_i) = \sum_{j=1}^{N} a_j\,\phi_{ij} \tag{16}$$

where

$$\phi_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \xi^2} \tag{17}$$

is the radial basis function known as multiquadrics. The constant ξ is the shape parameter. Although its value may affect the accuracy of the method, Chantasiriwan [3] showed that the method is relatively insensitive to the value of the shape parameter over a fairly large range. Eq. (16) is a component of the matrix equation,

$$\vec{f} = \Phi\vec{a} \tag{18}$$

which can be solved for the vector of coefficients.

$$\vec{a} = \Phi^{-1}\vec{f} \tag{19}$$

Once $\vec{a}$ has been determined, the approximation of a partial derivative of f with respect to x or y can be expressed in terms of functional values at all nodes. For example,

$$\frac{\partial\vec{f}}{\partial x} = \frac{\partial\Phi}{\partial x}\vec{a} \tag{20}$$

can be written as

$$\frac{\partial\vec{f}}{\partial x} = \left(\frac{\partial\Phi}{\partial x}\Phi^{-1}\right)\vec{f} \tag{21}$$

The first row of this matrix equation is thus the desired discretization of the partial derivative of f with respect to x at node 1. Therefore, this method can be used to express partial derivatives of f at any node i in terms of values of *f* at node i and N - 1 other nodes.

$$\left(\frac{\partial f}{\partial x}\right)_i = \sum a_{ij} f_j \qquad \textbf{(22)}$$

$$\left(\frac{\partial f}{\partial y}\right)_i = \sum b_{ij} f_j \qquad \textbf{(23)}$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_i = \sum c_{ij} f_j \qquad \textbf{(24)}$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_i = \sum d_{ij} f_j \qquad \textbf{(25)}$$

Right hand sides of Eqs. (22) - (25) are summations over all nodes *j* in the domain. It should be noted that, in each of Eqs. (22) - (25), most of coefficients in are zero except for N coefficients corresponding N selected nodes.

Discretization of Eqs. (13) - (15) at an interior node i using the local multiquadric collocation method and the implicit time-stepping scheme results in the following nonlinear algebraic equations:

$$\sum c_{ij}\psi_j^{(m)} + \sum d_{ij}\psi_j^{(m)} = -\omega_i^{(m)} \qquad \textbf{(26)}$$

$$\left(\frac{\omega_i^{(m)} + \omega_i^{(m\text{-}1)}}{\Delta t}\right) + \left(\sum b_{ij}\psi_j^{(m)}\right)\left(\sum a_{ij}\omega_j^{(m)}\right) - \left(\sum a_{ij}\psi_j^{(m)}\right)\left(\sum b_{ij}\omega_j^{(m)}\right) = \Pr\left(\sum c_{ij}\omega_j^{(m)} + \sum d_{ij}\omega_j^{(m)}\right) + \text{Ra}\Pr\sum a_{ij}T_j^{(m)} \qquad \textbf{(27)}$$

$$\left(\frac{T_i^{(m)} - T_i^{(m\text{-}1)}}{\Delta t}\right) + \left(\sum b_{ij}\psi_j^{(m)}\right)\left(\sum a_{ij}T_j^{(m)}\right) - \left(\sum a_{ij}\psi_j^{(m)}\right)\left(\sum b_{ij}T_j^{(m)}\right) = \sum c_{ij}T_j^{(m)} + \sum d_{ij}T_j^{(m)} \qquad \textbf{(28)}$$

where superscript m denotes time mΔt. In addition to Eqs. (26) - (28), algebraic equations resulting from discretization of boundary conditions for ψ, T and ω are needed. Usually, boundary conditions for ψ and T are given. The boundary condition for ω may be determined from velocity components according to Eq. (10). However, since boundary condition for an arbitrary boundary is usually specified in terms of velocity components normal and tangent to the boundary (u_n and u_t), it is more convenient to calculate boundary vorticity from

$$\omega = \frac{\partial u_n}{\partial t} - \frac{\partial u_t}{\partial n} \qquad \textbf{(29)}$$

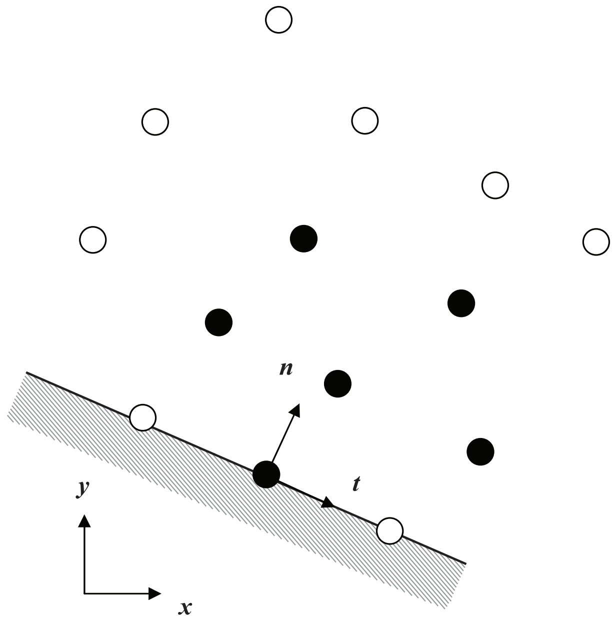

Fig. 1 Six nodes used for discretization of the vorticity boundary condition are the boundary node and 5 nearest interior nodes. All selected nodes are designated by solid circles.

If the boundary is impermeable, $u_n = 0$. For the purpose of discretizing Eq. (29), let node 1 be the boundary node where boundary vorticity is to be computed, and the other N – 1 be nearest interior nodes. As shown in Fig. 1, boundary nodes nearest to node 1 are excluded from this selection. If $\hat{n} = n_x \hat{x} + n_y \hat{y}$, the discretization of a partial derivative of $\vec{f}$ with respect to n may be written as

$$\frac{\partial \vec{f}}{\partial n} = \mathbf{A}\vec{f} \quad \textbf{(30)}$$

where

$$\mathbf{A} = \left(n_x \frac{\partial \Phi}{\partial x} + n_y \frac{\partial \Phi}{\partial y} \right) \Phi^{-1} \quad \textbf{(31)}$$

The expression for boundary vorticity at node 1 is then

$$\omega_1 = -A_{11} u_{t,1} - (A_{12} \quad A_{13} \quad \ldots \quad A_{1N}) . \begin{pmatrix} u_{t,2} \\ u_{t,3} \\ \vdots \\ u_{t,N} \end{pmatrix} \quad \textbf{(32)}$$

It can be shown that the relations and lead to $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ lead to

$$u_t = \frac{\partial \psi}{\partial n} \quad \textbf{(33)}$$

Eq. (33) allows the replacement of $u_{t,2}$, $u_{t,3}$, ..., $u_{t,N}$ at interior nodes in Eq. (32) with terms containing only stream function.

$$\omega_1 = -A_{11} u_{t,1} - (B_1 \quad B_2 \quad \ldots \quad B_N) . \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \tag{34}$$

where

$$(B_1 \quad B_2 \quad \ldots \quad B_N) = (A_{12} \quad A_{13} \quad \ldots \quad A_{1N}) . \begin{bmatrix} A_{21} & A_{22} & \cdots & A_{2N} \\ A_{31} & A_{32} & \cdots & A_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix} \tag{35}$$

The system of nonlinear equations formed by the above governing equations and boundary conditions must be solved by iteration. The iteration process starts with $\psi_i^{(0)} = T_i^{(0)} = \omega_i^{(0)} = 0$ at interior nodes. The successive over-relaxation method (SOR) is then used to find $\psi_i^{(1)}, T_i^{(1)}, \omega_i^{(1)}$. The iteration process is continued until convergence when the solution reaches the steady state. It was proven by Broyden [6] that this method is capable of giving converged solution if a suitable relaxation parameter is chosen.

4. Natural Convection in Square Cavity

A famous benchmark natural convection problem is the buoyancy-driven flow problem in square cavity. This problem is therefore used to test the performance of the local multiquadric collocation method. As shown in Fig. 2, the two horizontal sides of the cavity are insulated, and the two vertical sides are kept at two different temperatures. The length scale in this problem is the width of the cavity. Since the exact solution of this problem is not available, a numerical solution by the local multiquadric collocation method must be compared with the benchmark solution. Erturk and Gokcol [7] have obtained highly accurate solutions to the lid-driven flow problem by using the finite difference method and the vorticity boundary condition suggested by Stortkuhl et al. [8]. For the node arrangement shown in Fig. 3, boundary vorticity is computed as follows.

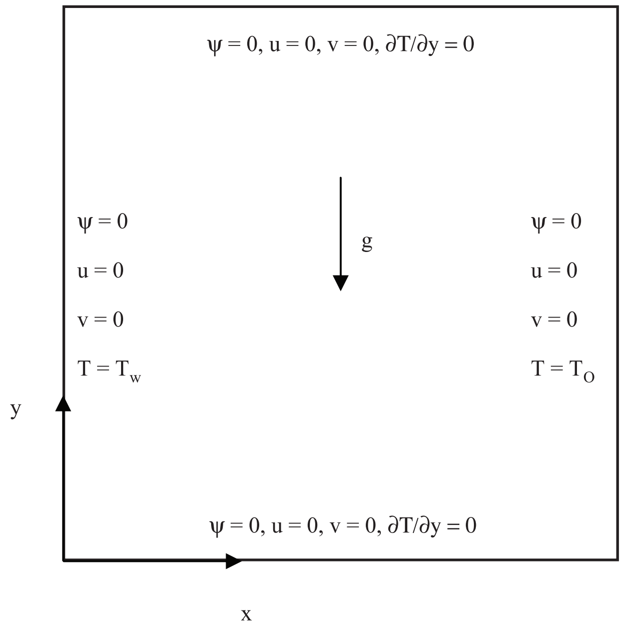

Fig. 2 Buoyancy-driven flow problem in a square cavity.

$$\omega_{k,0} = \frac{-9V}{2\Delta} - \frac{3(\psi_{k-1,1} + \psi_{k,1} + \psi_{k,1,1})}{2\Delta^2} - \frac{(\omega_{k-1,0} + \omega_{k+1,0})}{4} - \frac{(\omega_{k-1,1} + 4\;{}_{k+1,0} + \omega_{k+1,1})}{8} \quad \textbf{(36)}$$

where Δ is grid spacing, and V is the horizontal component of the boundary velocity. Note that $\psi_{k,0}$ is omitted in Eq. (36) because stream function is zero on the boundary. Expressions for boundary vorticity at other sides of the cavity can be similarly obtained. Fortran codes for solving to the lid-driven flow problem by using the finite difference method are available at http://www.cavityflow.com. These codes are modified by the author of this paper to solve the buoyancy-driven flow problem in a square cavity. Results on 161 × 161 square grid obtained for velocity components (u and v) and heat flux (q = $-\partial T/\partial x$) at selected points are considered to be benchmark solutions. Fig. 4 shows these results for Pr = 0.7 and Ra = 1000, 10000, and 100000.

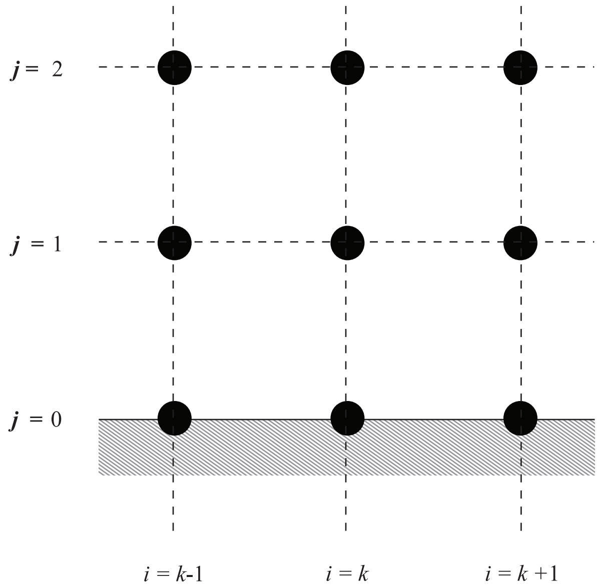

Fig. 3 Boundary nodes and nearby interior nodes at the bottom side of the square cavity.

The problem is then solved by the local multiquadric collocation method using the vorticity boundary condition corresponding to Eqs. (34) and (35). For the purpose of assessing the degree of accuracy of this method, it is useful to define error as the absolute value of the difference between the numerical solution by the local multiquadric collocation method (f) and the benchmark solution (f_e) divided by f_e:

$$\varepsilon f = \left| \frac{f - f_e}{f_e} \right| \quad \textbf{(37)}$$

where f represents u, v, or q. Errors of solutions by the local multiquadric collocation method are computed for Pr = 0.7 and Ra = 1000, 10000, and 100000. It can be seen from Fig. 5 that the local multiquadric collocation method is capable to producing fairly accurate solutions to this problem.

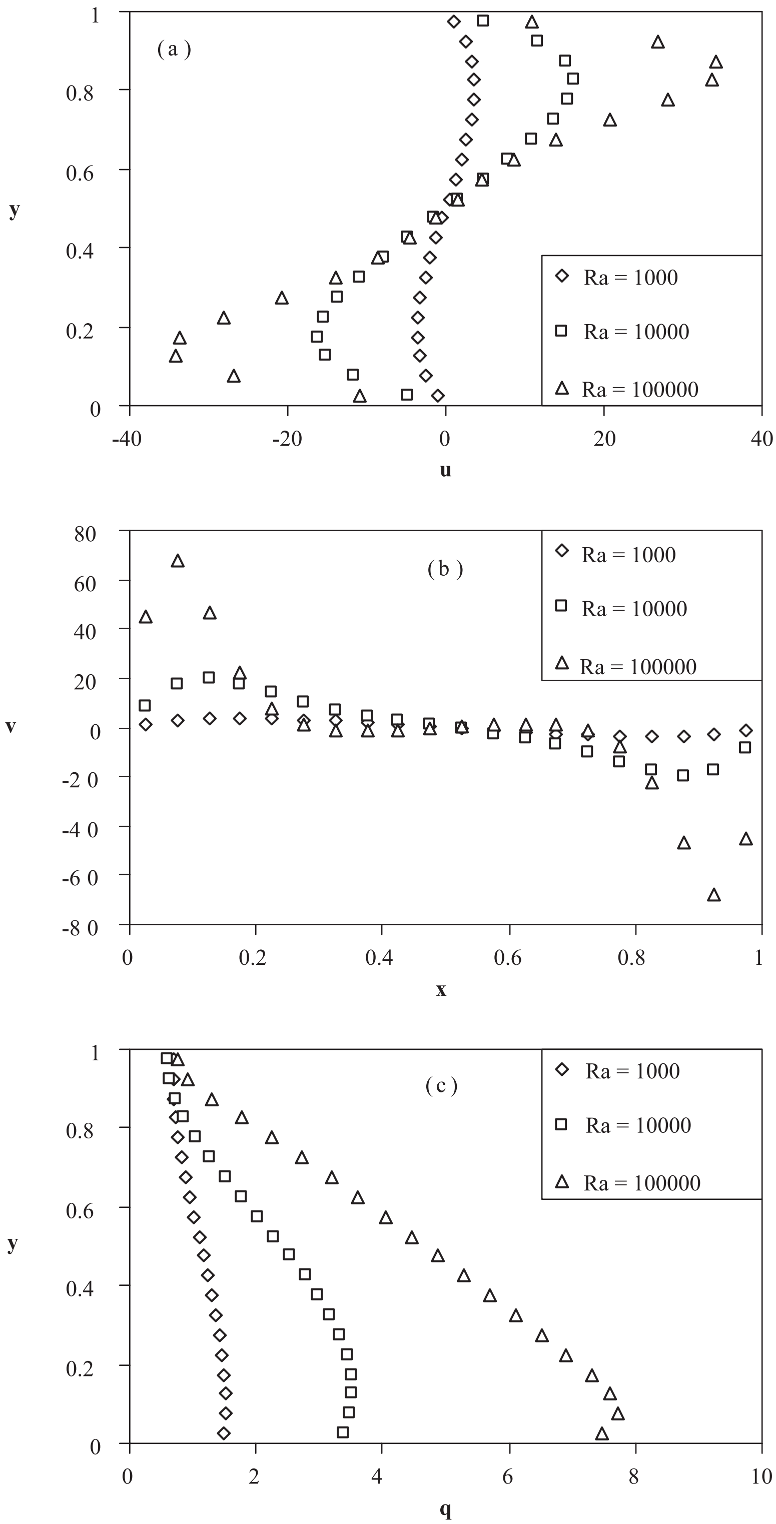

Fig. 4 Benchmark solutions for (a) the horizontal velocity component along the vertical line passing the center of the square cavity; (b) the vertical velocity component along the horizontal line passing the center of the cavity; and (c) the horizontal heat flux along the left wall of the cavity.

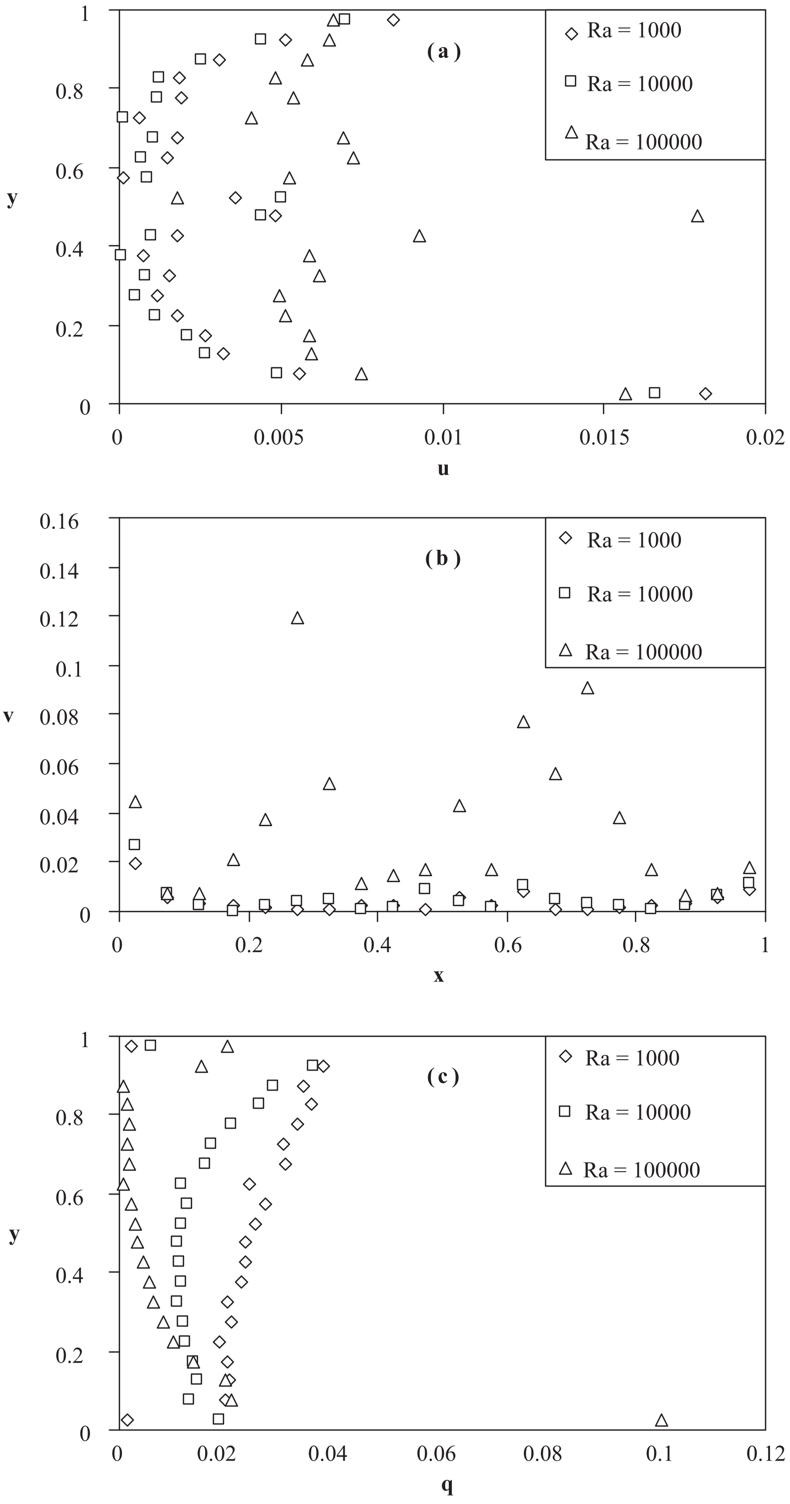

Fig. 5 Errors in (a) the horizontal velocity component; (b) the vertical velocity component; and (c) the horizontal heat flux of solutions by the local multiquadric collocation method.

5. Natural Convection in Horizontal Concentric Annulus

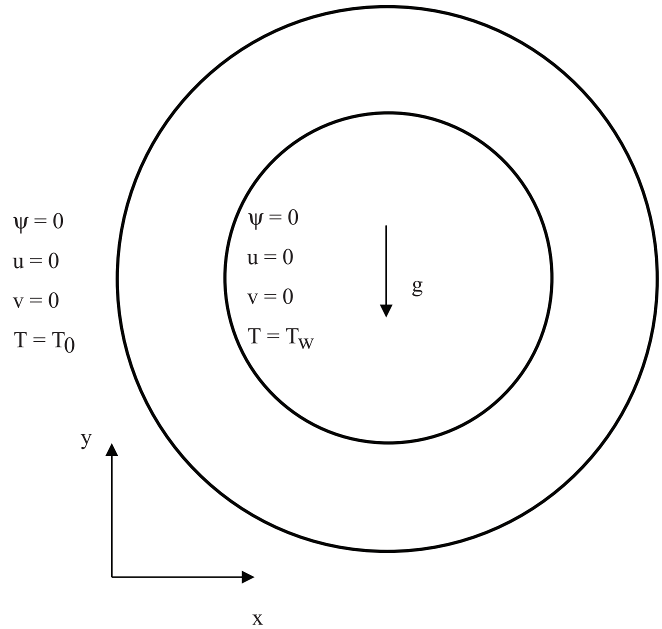

Fig. 6 Buoyancy-driven flow problem in a horizontal annulus.

Another natural convection problem that has received considerable attention in the literature is the natural convection problem annular space between two horizontal centric cylinders having infinite length. As shown in Fig. 6, the surfaces of the two cylinders are maintained at two different uniform temperatures, and both cylinders are stationary. This problem has been considered by Raithby and Hollands [9]. Their correlation based on the thickness of the annular space has been rewritten by Bejan [10] as a correlation based diameters of the inner and outer cylinders:

$$Q = \frac{2.425\, k(T_w - T_0)}{\left[1 + (D_i/D_o)^{3/5}\right]^{5/4}} \left(\frac{\mathrm{Pr}\, .\mathrm{Ra}_{\mathrm{Di}}}{0.861 + \mathrm{Pr}} \right)^{1/4} \quad (38)$$

where Q is heat transfer per unit length of the cylinders. Note that the length scale of the Rayleigh number in this correlation is the diameter of the inner cylinder (D_i). The expression of heat transfer by natural convection analogous to that of heat transfer by pure conduction is

$$Q = \frac{2\pi k_{eff}(T_w - T_0)}{\ln(D_o/D_i)} \qquad \textbf{(39)}$$

Thus, the ratio k_{eff}/k indicates the enhancement of heat transfer due to natural convection. The correlation in Eq. (39) is valid only when this ratio is larger than one.

This problem is solved by the local multiquadric collocation method. Instead of solving this problem in the cylindrical coordinates as done by some previous works, which necessitates the use of another set of governing equations, it is more convenient to solve this problem in the rectangular coordinates as given by Eqs. (13) - (15) with the length scale being the diameter of the inner cylinder. Fig. 7 compares numerical results for k_{eff}/k with experimental results as represented by correlation in Eq. (39), and shows that numerical results agree with experiment results.

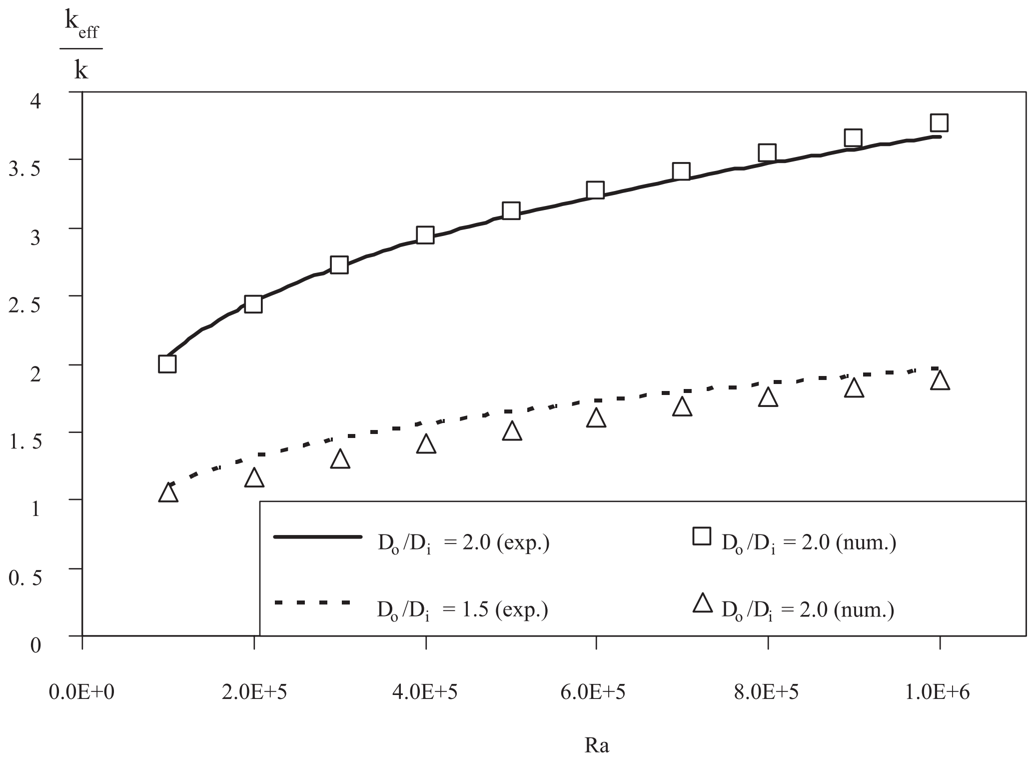

Fig. 7 Comparison between experimental and numerical values of k_{eff}/k.

6. Conclusions

The implementation of the local multiquadric collocation method for solving buoyancy-driven flow problems in the stream function-vorticity formulation is described in this paper. A scheme for imposing vorticity boundary condition is presented, which expresses boundary vorticity at a boundary node in terms of boundary velocity at the boundary node and values of stream function at nearby interior nodes. The local multiquadric collocation method is used to solve two buoyancy-driven flow problems. The first problem is the natural convection problem in a square cavity, of which two horizontal walls are insulated and two vertical walls are maintained at two different temperatures. The second problem is the natural convection in the annular space between two long concentric cylinders maintained at two different temperatures. It is shown that the performance of the method in solving both problems is good. Since the effectiveness of this method has been demonstrated, it should be considered as an alternative to conventional methods like the finite element method and the finite volume method for solving buoyancy-driven flow problems in an arbitrary geometry.

7. Acknowledgment

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