การสั่นอิสระของคานนาโนซึ่งพิจารณาผลของหน่วยแรงที่ผิวสัมผัส และลักษณะยืดหยุ่นแบบไม่เฉพาะที่

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บทคัดย่อ

งานวิจัยนี้มีวัตถุประสงค์เพื่อนำเสนอพฤติกรรมการสั่นแบบอิสระของคานนาโนภายใต้การเปลี่ยนแปลงเงื่อนไขของจุดรองรับ ซึ่งพิจารณาผลของหน่วยแรงที่ผิวสัมผัส (Surface Stress) และลักษณะยืดหยุ่นแบบไม่เฉพาะที่ (Non-local Elasticity) ร่วมกัน โดยใช้วิธีเชิงวิเคราะห์ในการแก้ปัญหาโดยตรงเพื่อให้ได้ค่าความถี่ธรรมชาติและรูปแบบการสั่น และตรวจสอบคำตอบด้วยวิธีเชิงตัวเลขโดยใช้ระเบียบวิธีไฟไนต์เอลิเมนต์ ผลการศึกษาแสดงให้เห็นว่าหน่วยแรงที่ผิวสัมผัสและลักษณะยืดหยุ่นแบบไม่เฉพาะที่ส่งผลโดยตรงต่อพฤติกรรมการสั่นอิสระของคานนาโน หน่วยแรงที่ผิวสัมผัสจะส่งผลให้สติฟเนสของคานนาโนสูงขึ้น ทำให้คานนาโนมีค่าความถี่ธรรมชาติสูงขึ้นเมื่อเปรียบเทียบผลจากทฤษฎีคานพื้นฐานในทุกเงื่อนไขของจุดรองรับ ซึ่งแสดงเห็นชัดในโหมดที่ต่ำของรูปแบบการสั่น ในขณะที่ผลของลักษณะยืดหยุ่นแบบไม่เฉพาะที่ทำให้คานนาโนมีค่าความถี่ธรรมชาติลดลง โดยเฉพาะรูปแบบการสั่นในโหมดที่สูงขึ้น สำหรับคานนาโนที่พิจารณาผลของหน่วยแรงที่ผิวสัมผัสและลักษณะยืดหยุ่นแบบไม่เฉพาะที่ร่วมกัน ค่าความถี่ธรรมชาติจะอยู่ระหว่างค่าของคานนาโนที่พิจารณาผลของหน่วยแรงที่ผิวสัมผัสและคานนาโนที่พิจารณาผลของลักษณะยืดหยุ่นแบบไม่เฉพาะที่เพียงอย่างเดียว เมื่อศึกษารูปแบบการสั่นของคานนาโน ผลของหน่วยแรงที่ผิวสัมผัสจะส่งผลต่อรูปแบบการสั่นในโหมดที่ต่ำเท่านั้น ในขณะที่รูปแบบการสั่นในโหมดที่สูงขึ้นจะใกล้เคียงกับกรณีคานนาโนที่พิจารณาผลของลักษณะยืดหยุ่นแบบไม่เฉพาะที่เพียงอย่างเดียว นอกจากนี้ผลของหน่วยแรงที่ผิวสัมผัสและลักษณะยืดหยุ่นแบบไม่เฉพาะที่จะไม่ส่งผลกระทบต่อรูปแบบการสั่นในเงื่อนไขของจุดรองรับแบบยึดหมุนทั้งสองด้าน (pinned-pinned) และจุดรองรับแบบด้านหนึ่งเลื่อนไถลได้ในแนวดิ่งกับอีกด้านหนึ่งยึดหมุน (sliding-pinned)

คำสำคัญ : คานนาโน / หน่วยแรงที่ผิวสัมผัส / ลักษณะยืดหยุ่นแบบไม่เฉพาะ / วิธีเชิงวิเคราะห์ / ระเบียบวิธีไฟไนต์เอลิเมนต์

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Surface stress and non-local elasticity effects on the free vibration behavior of nanobeams

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Abstract

This paper presents the effects of surface stress and non-local elasticity on the free vibration of nanobeams with various boundary conditions. Analytical solutions for natural frequencies and corresponding mode shapes of the nanobeams are derived and compared numerically by using the finite element method. Identical results between the two solution methods are obtained. The obtained results indicate that the surface stress and non-local elasticity affect directly the free vibration behavior of the nanobeams. The effect of surface stress is to increase the stiffness of the nanobeams. Therefore, the natural frequencies of the nanobeams for all boundary conditions increase in comparison with that of the classical Euler beam. Moreover, the surface stress affects significantly only for the lower modes of vibration. In the case of non-local elasticity effect, it reduces the natural frequencies of the nanobeams, especially for the higher modes of vibration. For the nanobeams subjecting to both effects, the results show that the natural frequencies are in between one of the nanobeams with surface stress and the one with non-local elasticity. This gives a practical implication that not only the surface stress but also non-local elasticity affects directly on the vibration behavior of the nanobeams. However, the surface stress affects only the lower modes of vibration and therefore the results of the nanobeams subjecting to both effects are converted to those of the nanobeams where only non-local elasticity is considered for the mode number with higher than fourth mode. In case of vibration mode shapes, the surface stress and non-local elasticity affect the nanobeams of all boundary conditions except the pinned-pinned and sliding-pinned nanobeams.

Keywords : Nanobeams / Surface Stress / Non-local Elasticity / Analytical Solution / Finite Element Method.

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1. Introduction

Nanoscale structures such as wires, rod and beams have a wide range of applications in physics, engineering, and several other fields [1-6]. In engineering applications, nanobeams are commonly used in advanced technological devices such as sensors, actuators, transistors, and resonators in nanoelectromechanical sys-tems (NEMSs), respectively [7,8]. Since the mechanical properties of nanobeams are totally different from classical and macroscopic counterparts, therefore it is necessary to exactly characterize the mechanical properties of nanobeams for these applications. Nanobeams are size-independent physical properties and mechanical properties while their bending, buckling, and vibration behaviors are size-dependent, this is due to their large ratio of surface area to volume [9,10].

For the solution of nanobeams obtained in literatures, it can be seen that most of the solutions of nanobeams were solved based on the consideration of surface stress, residual surface tension, as well as the surface elasticity. The surface stress theory was initiated by Gurtin and Murdoch [11] and was included by many researchers in order to investigate the static and dynamic behaviors of micro and nanostructures. For examples, Wang and Feng [3] presented the surface stress effect on the natural frequencies of simply-supported micro-beams. The surface stress effect was also considered incorporated with nonlinear static and dynamic behaviors of nanobeams [4]. In addition, Liu and Rajapzikse [5] studied the surface stress effect on the bending, buckling, and the vibration of nanobeams and double nanobeams systems [6].

In case of non-local elasticity theory, the obtained literatures indicated that the non-local elasticity theory was initially proposed in 1983 by Eringen [12,13]. Later on, this theory was applied to the classical nanostructure of carbon nanotubes (CNTs). Based on this theory, the size scale is used to adjust the properties of elastic body continuum and it is assumed that the stress at a reference point depends on the strain at the same point and also at all other points in the continuum. In this way, the internal size scale could be considered in the constitutive equations, especially, the size scale of material parameters.

The applications of non-local continuum theory on the nanotechnology works were initially addressed by Peddieson et al. [14]. This work used the simplified non-local model which presented by Eringen [12] to solve the static displacement of nanostructures. In literatures, there were also found that applications of non-local elasticity for nanostructure analysis received much attention by many researchers [15-21]. Reddy and Pang [15] presented the analytical solutions of bending, vibration, and buckling of CNTs. Non-local elasticity for free vibration single-walled CNTs are also presented by author's previous work [16]. This work presented the classical solutions and then compared with that of numerical solutions provided by finite element models. Lu et al. [17] studied the dynamic properties of flexural beams using a non-local elasticity model. Frequency equations and model shape functions of nanobeams with general boundary conditions were derived based on a non-local Euler-beam model. Pandikar and Pradhan [18] analyzed the bending, buckling, and the vibration of nanobeams and nanoplates by using the finite element solution. Moreover, a few researchers [19,20] were also applied the non-local elasticity with the free vibration behaviors of nanobeams using finite element method. Meanwhile, Ghannadpour et al. [21] presented bending, buckling, and the vibration problems of non-local Euler beam using Ritz method.

Focusing on the problem of nanobeams including the combined effects of surface stress and non-local elasticity, Mahmoud et. al. [22] presented the static displacement of simply support nanobeams by using finite element analysis while author's previous work [23] determined analytically the static displacement and buckling of nanowires with various boundary conditions and verified numerically using finite element method. Few studies included the surface stress effect on the frequency analysis of nanostructures using non-local elasticity beam theory were also found in works of Lee and Chang [24,25]. The first published paper presented the natural frequency of simply supported nanotubes using the non-local Timoshenko beam theory while the second work presented the natural frequency of a non-uniform cantilever nanobeam using the Raleigh-Ritz approximation solution method. In addition, the nonlinear frequency analysis of nonuniform cross section nanobeams was also investigated by Malekzadeh and Shojaee [26] for both Timoshenko and Euler-Bernoulli beam theories.

There can be seen from literatures that only a few studies have focused on vibration behaviors of nanostructures including the combined effects of surface stress and non-local elasticity as well as the combined effects on the vibration mode shapes of nanostructures have not been found yet. Therefore, the main purpose of this work is to determine analytically the natural frequencies and corresponding mode shapes of nanobeams with boundary conditions of pinned-pinned, clamped-clamped, clamped-free, clamped-pinned, clamped-sliding, and sliding-pinned, respecttively, and then verified numerically using the finite element method. The combined effects of surface stress and non-local elasticity on the natural frequencies, vibration mode shapes, and boundary conditions are also discussed and would differentiate this work from other published works [24,25]. The equation of motion of nanobeams and the free vibration analysis results obtained in this work can be used to predict the vibration behaviors of nanobeams in NEMs technology applications.

2. Problem Formulation

2.1 Surface stress model for nanobeams

The influence of surfaces can be expressed as surface stress which is related to the surface density. In the theory of surface elasticity [11] if both the surface layer and the bulk of the material are isotropic and linearly elastic, the one-dimensional form of surface stress can be given by [1,2]

$$\sigma^s = \tau^0 + E^s \varepsilon \qquad \textbf{(1)}$$

where τ^0 is the residual surface tension, is the surface elasticity determining by atomistic simulations or the experimental measurements [9,10], and ε is the longitudinal strain of the surface resulting from an applied force to the nanobeams.

The assumption that the thickness of the surface layer t is much smaller than the beam thickness is applied. The effect of surface elasticity on the vibration of nanobeams can be accounted by replacing the flexural rigidity (EI) for the bulk material with that of the effective flexural rigidity $(EI)^*$ [1,2] and can be presented as

$$(EI)^* = \begin{cases} E\dfrac{bh^3}{12} + E^s\left(\dfrac{bh^2}{2} + \dfrac{h^3}{6}\right) & (rectangular) \\ E\dfrac{\pi D^4}{64} + E^s\dfrac{\pi D^3}{8} & (circular) \end{cases} \qquad \textbf{(2)}$$

where b and h are the width and depth of rectangular cross section while D is the diameter of circular cross section as presented in Fig. 1.

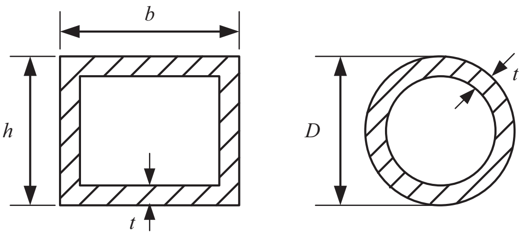

Fig. 1 A rectangular and circular cross section of nanobeams with a surface layer.

According to the generalized Young-Laplace equation [1,2], the distributed transverse force q_n resulting from the surface stress along nanobeams in longitudinal direction and depending on the current surface curvature can be obtained as

$$q_n = H\kappa \quad \textbf{(3)}$$

where κ is the surface curvature approximated by d^2v/dx^2 for small deformation of nanobeams in which v is the transverse displacement of nanobeams. The parameter H is a constant determined by the residual surface tension depending on the shape of cross section. For a rectangular cross section with the width of b and a circular cross section with the diameter of D, the parameter H is given by

$$H = \begin{cases} 2\tau^0 b & (rectangular) \\ 2\tau^0 D & (circular) \end{cases} \quad \textbf{(4)}$$

2.2 Surface stress including non-local elasticity model for nanobeams

According to the theory of non-local elasticity, stress at a reference point depends on the strain at the same point and also at all other points in the continuum. The non-local bending moment constitutive relations for one-dimensional of nanowires including the effects of non-local elasticity [12] and surface stress can be expressed as

$$M = -(EI)^* \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 M}{\partial x^2} \quad \textbf{(5)}$$

where $\mu = (e_0 a)^2$ is the parameter of non-local scale revealing the effect of small-scale on the response of nanostructures while $(EI)^*$ is the effective flexural rigidity including surface stress mentioned above. Besides, Yang and Lim [27] estimated the values of $\overline{\mu}$ as $\overline{\mu} \leq 0.04$ by matching the analytical non-local parameter of Timoshenko nanobeams model and molecular dynamic simulation solutions. The value of $\overline{\mu}$ is the dimensionless non-local parameter which $\overline{\mu} = \mu/L^2$ where L is the total length of nanobeams.

From the equilibrium of forces and moments on the nanobeams segment as shown in Fig. 2, the equations of shear force Q and the bending moment M can be written as

$$\frac{\partial Q}{\partial x} = \rho A \frac{\partial^2 v}{\partial t^2} - H \frac{\partial^2 v}{\partial x^2} \quad \textbf{(6)}$$

$$\frac{\partial M}{\partial x} = Q \quad \textbf{(7)}$$

where ρ is the mass density and A is the cross section area of the nanobeams, respectively. Substituting of Eq. (6) into Eq. (7), the following equation can be obtained

$$\frac{\partial^2 M}{\partial x^2} = \rho A \frac{\partial^2 v}{\partial t^2} - H \frac{\partial^2 v}{\partial x^2} \quad \textbf{(8)}$$

Substituting Eq. (8) into Eq. (5), one obtains

$$M = -(EI)^* \frac{\partial^2 v}{\partial x^2} + \mu \left\{ \rho A \frac{\partial^2 v}{\partial t^2} - H \frac{\partial^2 v}{\partial x^2} \right\} \quad \textbf{(9)}$$

By setting Eq. (8) equal to the second derivative of Eq. (9), the equation of motion for the nanobeams with consideration of both surface stress and non-local elasticity can be expressed as

$$\left[(EI)^{*}+\mu H\right]\frac{\partial^4 v}{\partial x^4}-H\frac{\partial^2 v}{\partial x^2}-\mu\rho A\frac{\partial^4 v}{\partial x^2\partial t^2}+\rho A\frac{\partial^2 v}{\partial t^2}=0 \quad \textbf{(10)}$$

In case of the surface stress and the non-local elasticity effects are completely neglected which E_s, τ^0, H and are μ all set to zero, Eq. (10) can be reduced to the classical Euler beam equation [28].

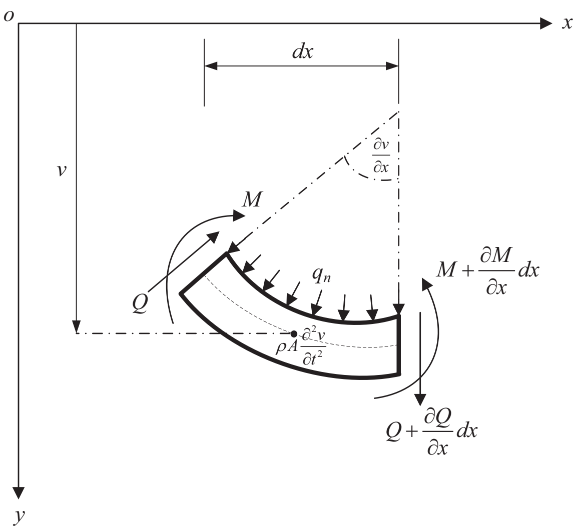

Fig. 2 Free body of an infinitesimal segment of nanobeams.

3. Analytical solution for free vibration behaviors of nanobeams

For determining the natural frequencies and corresponding mode shapes, the solution of partial differential equation of motion, Eq. (10), can be obtained by assuming the displacement of the nanobeams as

$$v(x,t)=V(x)e^{i\omega t},\; i=\sqrt{-1} \quad \textbf{(11)}$$

where ω is the eigenvalue and $V(x)$ is the eigenfuction, respectively. Substituting Eq. (11) into Eq. (10), the governing equation for transverse free vibration of nanobeams considering surface stress and non-local elasticity can be obtained as

$$\left[(EI)^{*}+\mu H\right]\frac{d^4 V}{dx^4}-\left(H-\mu\rho A\omega^2\right)\frac{d^2 V}{dx^2}-\rho A\omega^2 V=0 \quad \textbf{(12)}$$

The bending stiffness of nanobeams can be defined by using the parameter α where α = [$(EI)^{*}$ + (μH)]. For convenience, the following dimensionless parameters are introduced:

$$\xi=x/L,\; \bar{\mu}=\mu/L^2,\; \eta=\frac{(H-\mu\rho A\omega^2)L^2}{\alpha}$$

and $\beta=\dfrac{\rho A\omega^2 L^4}{\alpha}$. **(13a-13d)**

Therefore, the governing equations for transverse free vibration of nanobeams can be simply expressed in dimensionless form as

$$\frac{d^4 V}{d\xi^4}-\eta\frac{d^2 V}{d\xi^2}-\beta V=0 \quad \textbf{(14)}$$

Finally, the solution of Eq. (14) can be expressed as

$$V(\xi)=C_1\sinh\lambda\xi+C_2\cosh\lambda\xi+C_3\sin\gamma\xi+C_4\cos\gamma\xi \quad (15)$$

where $\lambda^2=\frac{1}{2}(\eta+\sqrt{\eta^2+4\beta^2})$, $\gamma^2=\frac{1}{2}(-\eta+\sqrt{\eta^2+4\beta^2})$ and the unknown constants C_1, C_2, C_3 and C_4 for each case of the support condition can be determined below.

3.1 Pinned-Pinned nanobeams

The boundary conditions for pinned-pinned nanobeams are given as $V(0) = V(1) = 0$ and $M(0) = M(1) = 0$. By substituting the general solution of nanobeams, Eq. (15), into the boundary conditions, the characteristic equation can be obtained as

$$\sin(\gamma) = 0 \tag{16}$$

Then, the unknown constants for pinned-pinned nanobeams are given as

$$C_1 = 0, \tag{17a}$$
$$C_2 = 0, \tag{17b}$$
$$C_3 = 0, \tag{17c}$$
and
$$C_1 = 0. \tag{17d}$$

Next, the eigenvalue of natural frequencies is also obtained in form of

$$\omega^2 = \frac{\alpha(n\pi / L)^4 + H(n\pi / L)^2}{\rho A(1+\bar{\mu}(n\pi)^2)}, \ n = 1,2,3,... \tag{18}$$

Finally, the corresponding modal shape function is

$$V(\xi) = \sin(\gamma\xi) \tag{19}$$

3.2 Clamped-Clamped nanobeams

The boundary conditions of clamped-clamped nanobeams are giv en as $V(0) = V'(0) = 0$ and $V(1) = V'(1) = 0$ which are the zero displacement and rotation at beam's support. Then, the characteristic equation for determining the natural frequencies can be obtained as

$$\beta^2 + \beta\eta \sinh\lambda \sin\gamma + (2\eta^2 + \beta^2) \tag{20}$$

Then, the unknown constants for clamped-clamped nanobeams are given as

$$C_1 = 1, \tag{21a}$$

$$C_2 = \frac{\lambda\sin\gamma - \gamma\sinh\lambda}{\gamma(\cosh\lambda - \cos\gamma)}, \tag{21b}$$

$$C_3 = -\frac{\lambda}{\gamma}, \tag{21c}$$

and

$$C_4 = -\frac{\lambda\sin\gamma - \gamma\sinh\lambda}{\gamma(\cosh\lambda - \cos\gamma)}. \tag{21d}$$

Finally, the modal shape function of clamped-clamped nanobeams is

$$V(\xi) = \sinh(\lambda\xi) + \frac{\lambda\sin\gamma - \gamma\sinh\lambda}{\gamma(\cosh\lambda - \cos\gamma)}\cosh(\lambda\xi) - \frac{\lambda}{\gamma}\sin(\gamma\xi) - \frac{\lambda\sin\gamma - \gamma\sinh\lambda}{\gamma(\cosh\lambda - \cos\gamma)}\cos(\gamma\xi) \tag{22}$$

3.3 Clamped-Free nanobeams

The boundary conditions for clamped-free nanobeams are given as $\bar{v}(0) = \bar{v}'(0) = 0$ and $M(1) = Q(1) = 0$ in which they are corresponding to the displacement and rotation at clamp end to be zero while the moment and shear at free end to be vanished, respectively. Next, the characteristic equation for determining the natural frequencies is obtained as follow:

$$\beta^2 + \beta\eta\sinh\lambda\sin\gamma + (2\eta^2 + \beta^2)\cosh\lambda\cos\gamma = 0 \tag{23}$$

Then, the unknown constants for clamped-free nanobeams can be expressed as

$$C_1 = 1, \quad \textbf{(24a)}$$

$$C_2 = -\frac{\lambda^2 \sinh\lambda + \lambda\gamma \sin\gamma}{\lambda^2 \cosh\lambda + \gamma^2 \cos\gamma}, \quad \textbf{(24b)}$$

$$C_3 = -\frac{\lambda}{\gamma}, \quad \textbf{(24c)}$$

and

$$C_4 = \frac{\lambda^2 \sinh\lambda + \lambda\gamma \sin\gamma}{\lambda^2 \cosh\lambda + \gamma^2 \cos\gamma}. \quad \textbf{(24d)}$$

Finally, the modal shape function of clamped-free nanobeams is given as follow:

$$V(\xi) = \sinh(\lambda\xi) - \frac{\lambda^2 \sinh\lambda + \lambda\gamma \sin\gamma}{\lambda^2 \cosh\lambda + \gamma^2 \cos\gamma}\cosh(\lambda\xi) - \frac{\lambda}{\gamma}\sin(\gamma\xi) + \frac{\lambda^2 \sinh\lambda + \lambda\gamma \sin\gamma}{\lambda^2 \cosh\lambda + \gamma^2 \cos\gamma}\cos(\gamma\xi) \quad \textbf{(25)}$$

3.4 Clamped-Pinned nanobeams

The boundary conditions for clamp-pinned nanobeams are given as $V(0) = V'(0) = V(1)$ and $M(1) = 0$. Next, the characteristic equation for determining the natural frequencies is obtained as

$$\lambda \cosh\lambda \sin\gamma - \gamma \sinh\lambda \cos\gamma = 0 \quad \textbf{(26)}$$

Then, the unknown constants for clamped-pinned nanobeams are obtained as

$$C_1 = 0, \quad \textbf{(27a)}$$

$$C_2 = -\tanh\lambda, \quad \textbf{(27b)}$$

$$C_3 = \frac{\lambda}{\gamma}, \quad \textbf{(27c)}$$

and

$$C_4 = \frac{\lambda}{\gamma}\tanh\gamma. \quad \textbf{(27d)}$$

Finally, the modal shape function of clamped-free nanobeams is given as follow:

$$V(\xi) = \sinh(\lambda\xi) - \tanh\lambda\cosh(\lambda\xi) - \frac{\lambda}{\gamma}\sin(\gamma\xi) + \frac{\lambda}{\gamma}\tanh\lambda\cos(\gamma\xi) \quad \textbf{(28)}$$

3.5 Clamped-Sliding nanobeams

The boundary conditions for clamp-sliding nanobeams are given as $V(0) = V'(0) = 0$ and $V'(1) = Q(1) = 0$. Next, the characteristic equation for determining the natural frequencies is obtained as

$$\gamma \cosh\lambda \sin\gamma + \lambda \sinh\lambda \cos\gamma = 0 \quad \textbf{(29)}$$

Then, the unknown constants for clamped-sliding nanobeams are obtained as

$$C_1 = 1, \quad \textbf{(30a)}$$

$$C_2 = -\frac{\lambda(\cosh\lambda - \cos\gamma)}{\lambda\sinh\lambda + \gamma\sin\gamma}, \quad \textbf{(30b)}$$

$$C_3 = \frac{\lambda}{\gamma}, \quad \textbf{(30c)}$$

and

$$C_4 = \frac{\lambda(\cosh\lambda - \cos\gamma)}{\lambda\sinh\lambda + \gamma\sin\gamma}. \quad \textbf{(30d)}$$

Finally, the modal shape function of clamped-free nanobeams can be expressed as

$$V(\xi) = \sinh(\lambda\xi) - \frac{\lambda(\cosh\lambda - \cos\gamma)}{\lambda\sinh\lambda + \gamma\sin\gamma}\cosh(\lambda\xi) - \frac{\lambda}{\gamma}\sin(\gamma\xi) + \frac{\lambda(\cosh\lambda - \cos\gamma)}{\lambda\sinh\lambda + \gamma\sin\gamma}\cos(\gamma\xi) \quad \textbf{(31)}$$

3.6 Sliding-Pinned nanobeams

The boundary conditions for sliding-pinned nanobeams are given as $V'(0) = Q(0) = 0$ and $M(1) = 0$. Next, the characteristic equation for determining the natural frequencies is obtain as

$$\cos(\gamma) = 0 \quad \textbf{(32)}$$

Then, the unknown constants for sliding-pinned nanobeams are obtained as

$$C_1 = 0, \quad (33a)$$
$$C_2 = 0, \quad (33b)$$
$$C_3 = 0, \quad (33c)$$
and
$$C_4 = 1. \quad (33d)$$

Finally, the modal shape function of sliding-pinned nanobeams is

$$V(\xi) = \cos(\gamma\xi) \quad \textbf{(34)}$$

To check the validity of natural frequencies and vibration mode shapes of nanobeams, the Galerkin finite element method is used to verify these results. A brief detail of the method is given in the next section.

4. Finite element solution for vibration of nanobeams

By applying the Galerkin's weighted residual method to Eq. (12), the weighted residual equation can be written as

$$\int_0^L \left\{ \begin{array}{l} \left[(EI)^* + \mu H)\right]\dfrac{d^4V}{dx^4} \\ -\left(H - \mu\rho A\omega^2\right)\dfrac{d^2V}{dx^2} \\ -\rho A\omega^2 V \end{array} \right\} \bar{v}(x)dx = 0 \quad \textbf{(35)}$$

where L is the total length of nanobeams and $\bar{v}(x)$ is the weight functions. Integrating by parts of Eq. (35) twice, one yields the following equation:

$$\int_0^L \left\{ \begin{array}{l} \left[(EI)^* + \mu H\right]\dfrac{d^2V}{dx^2}\dfrac{d^2\bar{v}}{dx^2} \\ +(H - \mu\rho A\omega^2)\dfrac{dV}{dx}\dfrac{d\bar{v}}{dx} - \rho A\omega^2 V\bar{v} \end{array} \right\} dx + \left(M\dfrac{d\bar{v}}{dx} - Q\bar{v} \right)\Bigg|_0^L = 0 \quad \textbf{(36)}$$

where $M = -\left[(EI)^* + \mu H\right]\dfrac{d^2V}{dx^2} - \mu\rho A\omega^2 V$ and $Q = -\left[(EI)^* + \mu H\right]\dfrac{d^3V}{dx^3} - \mu\rho A\omega^2\dfrac{dV}{dx} + H\dfrac{dV}{dx}$ are the bending moment and shear force terms, respectively.

For two-node finite element with two nodal degrees of freedom per node, the transverse displacement is interpolated in terms of shape functions and degrees of freedom as

$$\bar{v}(x) = [\mathbf{N}]\{\mathbf{d}\} \quad \textbf{(37)}$$

where $\{\mathbf{d}\} = \left[V_1 \quad \dfrac{dV_1}{dx} \quad V_2 \quad \dfrac{dV_2}{dx} \right]^T$ is the nodal degree of freedom and $\mathbf{N}_i(x)$ is the cubic polynomial shape functions which can be given as

$$\mathrm{N}_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}, \quad \textbf{(38a)}$$

$$\mathrm{N}_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}, \quad \textbf{(38b)}$$

$$\mathrm{N}_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}, \quad \textbf{(38c)}$$

and

$$\mathrm{N}_4 = \frac{x^3}{l^2} - \frac{x^2}{l}. \quad \textbf{(38d)}$$

where l is the length of nanobeams-element. Substitution of Eq. (37) into Eq. (36), one yields

$$\int_0^l \begin{Bmatrix} [(EI)^* + \mu H][\mathbf{N}'']^T[\mathbf{N}''] \\ +H[\mathbf{N}']^T[\mathbf{N}'] \\ -\omega^2(\rho A[\mathbf{N}]^T[\mathbf{N}] \\ +\mu\rho A[\mathbf{N}']^T[\mathbf{N}']) \end{Bmatrix} \{\mathbf{d}\} dx = 0 \quad \textbf{(39)}$$

The element stiffness matrix is then obtained as

$$[\mathbf{k_e}] = \int_0^l \begin{Bmatrix} [(EI)^* + \mu H][\mathbf{N}'']^T[\mathbf{N}''] \\ +H[\mathbf{N}']^T[\mathbf{N}'] \end{Bmatrix} dx \quad \textbf{(40a)}$$

, or

$$[\mathbf{k_e}] = \frac{[(EI)^* + \mu H]}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} + \frac{H}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \quad \textbf{(40b)}$$

It can be seen from Eq. (40b) that the modified stiffness presents classical stiffness matrix including the both effects of surface stress and non-local elasticity. Then, the element mass matrix is given by

$$[\mathbf{m_e}] = \int_0^l \begin{Bmatrix} \rho A[\mathbf{N}]^T[\mathbf{N}] \\ +\mu\rho A[\mathbf{N}']^T[\mathbf{N}'] \end{Bmatrix} dx \quad \textbf{(41a)}$$

, or

$$[\mathbf{m_e}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} + \frac{\mu\rho A}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \quad \textbf{(41b)}$$

Finally, the element stiffness matrices, mass matrices, and nodal degree of freedom are assembled to obtain the global equilibrium equation of motion as

$$([\mathbf{K}] - \omega^2\ [\mathbf{M}])\ \{\mathbf{D}\} = \{\mathbf{0}\} \quad \textbf{(42)}$$

where $[\mathbf{K}] = \sum_{i=1}^{N_{els}} [\mathbf{k_e}]_i$, $[\mathbf{M}] = \sum_{i=1}^{N_{els}} [\mathbf{m_e}]_i$, and $\{\mathbf{D}\} = \sum_{i=1}^{N_{els}} \{\mathbf{d}\}_i$, are the global structural stiffness matrix, global consistent mass matrix, global nodal displacement, respectively. N_{els} is the number of elements in the structure. Then, Eq. (42) has the form of the algebraic eigenvalue problem. For non-trivial solution, the determinant of the coefficient matrix is equal to zero.

$$\left|[\mathbf{K}] - \omega^2\ [\mathbf{M}]\right| = 0 \quad \textbf{(43)}$$

By expansion of Eq. (43), one yields a polynomial of order n of characteristic equation. The n roots of ω_i^2 are the eigenvalue which it is ordered from lowest to highest mode of vibration for each boundary condition.

$$0 \le \omega_1^2 \le \omega_2^2 \le \ldots \le \omega_i^2 \le \ldots \omega_n^2 \quad \textbf{(44)}$$

5. Results and discussion

In this section, the results of natural frequencies and corresponding mode shapes of nanobeams with various boundary conditions are presented using the nanobeams with the span length (L) of 1,000 nm and the diameter (D) of 50 nm as the case study. The material properties as used in the work of He and Lilley [1,2] are performed. These material properties are E = 76 GPa, E^s = 1.22 N/m, τ^0 = 0.89 N/m, and ρ = 10.5×10^3 kg/m^3, respectively. In case of non-local parameter, the authors use the parameter presented in the work of Yang and Lim [27] which $\overline{\mu}$ = 0.04.

The analytical results for natural frequencies of nanobeams are verified with that of finite element analysis results by using 20 elements discretized along the beam's span length. As presented in Tables1-2, the analytical and numerical results for all boundary conditions of nanobeams are identical for the nanobeams with classical Euler beam theory, nanobeams with non-local elasticity, nanobeams with surface stress, and nanobeams with combined effects of non-local elasticity and surface stress. Moreover, the obtained results also demonstrate high accuracy in comparison with those of previous analysis results [2, 17].

Table 1 Analytical and numerical results of eigenvalue, ω(x10^9) of pinned-pinned (P-P), clamped-clamped (C-C), and clamped-free (C-F) nanowires

Boundary condition		Classical Euler beam		Surface stress effect		Non-local elasticity effect		Surface and Non-local effects	
		LT[a] [28]	FEM[b]	LT[a] [2]	FEM[b]	LT[a] [17]	FEM[b]	LT[a]	FEM[b]
P-P	Mode 1	0.3319	0.3319	0.3912	0.3912	0.2810	0.2810	0.3489	0.3489
	Mode 2	1.3276	1.3276	1.3919	1.3919	0.8267	0.8267	0.9249	0.9249
	Mode 3	2.9872	2.9873	3.0545	3.0545	1.3999	1.3999	1.5324	1.5325
	Mode 4	5.3106	5.3111	5.3811	5.3817	1.9633	1.9635	2.1322	2.1324
C-C	Mode 1	0.7524	0.7524	0.7877	0.7877	0.6151	0.6151	0.6878	0.6878
	Mode 2	2.0740	2.0741	2.1240	2.1240	1.2249	1.2249	1.3431	1.3431
	Mode 3	4.0660	4.0662	4.1233	4.1235	1.8336	1.8338	1.9922	1.9923
	Mode 4	6.7212	6.7223	6.7846	6.7858	2.4083	2.4088	2.6065	2.6070
C-F	Mode 1	0.1182	0.1182	0.1801	0.1801	0.1085	0.1085	0.1728	0.1728
	Mode 2	0.7410	0.7410	0.8301	0.8301	0.4902	0.4902	0.5806	0.5806
	Mode 3	2.0748	2.0749	2.1563	2.1563	1.0553	1.0553	1.1671	1.1671
	Mode 4	4.0659	4.0660	4.1462	4.1464	1.6234	1.6235	1.7698	1.7698

LT[a] = Linear Beam Theory (Analytical solution)
FEM[b] = Finite Element Method

Table 2 Analytical and numerical results of eigenvalue,of clamped-pinned (C-P), clamped-sliding (C-S), and sliding-pinned (S-P) nanowires

Boundary condition		Classical Euler beam		Surface stress effect		Non-local elasticity effect		Non-local and Surface effects	
		LT[a] [28]	FEM[b]	LT[a]	FEM[b]	LT[a]	FEM[b]	LT[a]	FEM[b]
C-P	Mode 1	0.5185	0.5185	0.5649	0.5649	0.4286	0.4286	0.4965	0.4965
	Mode 2	1.6803	1.6803	1.7366	1.7366	1.0202	1.0202	1.1271	1.1271
	Mode 3	3.5058	3.5059	3.5677	3.5678	1.6105	1.6101	1.7554	1.7554
	Mode 4	5.9951	5.9960	6.0618	6.0626	2.1832	2.1836	2.3663	2.3667
C-S	Mode 1	0.1881	0.1881	0.2205	0.2205	0.1775	0.1775	0.2195	0.2195
	Mode 2	1.0165	1.0165	1.0689	1.0689	0.7206	0.7206	0.8084	0.8084
	Mode 3	2.5101	2.5101	2.5693	2.5694	1.3178	1.3178	1.4436	1.4436
	Mode 4	4.6675	4.6679	4.7318	4.7321	1.8989	1.8991	2.0628	2.0629
S-P	Mode 1	0.0083	0.0083	0.1325	0.1325	0.0079	0.0079	0.1301	0.1301
	Mode 2	0.7468	0.7468	0.8093	0.8093	0.5434	0.5435	0.6261	0.6261
	Mode 3	2.0744	2.0745	2.1402	2.1403	1.1140	1.1140	1.2290	1.2290
	Mode 4	4.0659	4.0662	4.1347	4.1349	1.6830	1.6831	1.8335	1.8336

LT[a] = Linear Beam Theory (Analytical solution)
FEM[b] = Finite Element Method

In this investigation, the variations of mode shapes of nanobeams for all boundary conditions are also presented. The variations of mode shapes of pinned-pinned nanobeams are shown in Fig. 3. It is interesting to note that the non-local elasticity and surface stress have no effect on the vibration mode shapes since the modal shape function, Eq. (19), of pinned-pinned nanobeams is independent from both effects. In other words, the non-local elasticity and surface stress effects affect only on natural frequencies in case of pinned-pinned nanobeams.

In Figs. 4-7, the variations of mode shapes of clamped-clamped, clamped-free, clamped-pinned and clamped-sliding nano-beams are presented. It can be seen that surface stress and non-local elasticity terms affects directly on the vibration mode shapes of nanobeams. The surface stress increases the vibration amplitudes. The influence of this effect is also described by considering the modified stiffness of Eq. (40), which the vibration amplitude basically increases when the stiffness matrix increases. Consequently, the non-local elasticity decreases the vibration amplitudes of nanobeams. This is due to the effect of non-local elasticity term in the mass matrix of Eq. (41). When both effects are combined, the similar results of vibration mode shapes with that of the nanobeams including only the non-local elasticity are obtained.

This gives a practical implication that the surface stress has no significantly effect on the vibration amplitude.

The mode shapes of sliding-pinned nanobeams is presented in Fig. 8 and also found that the results of nanobeams including both effects of surface stress and non-local elasticity is not different from the Euler beam.

In the other words, the sliding-pinned nanobeams gives similar vibration behavior as pinned-pinned nanobeams. The boundary condition can be satisfied by the following vibration of cosh $\gamma = 0$ or $\gamma = (2n-1)\ \pi/2$ and the mode number is independent on both non-local elasticity and Surface stress effects.

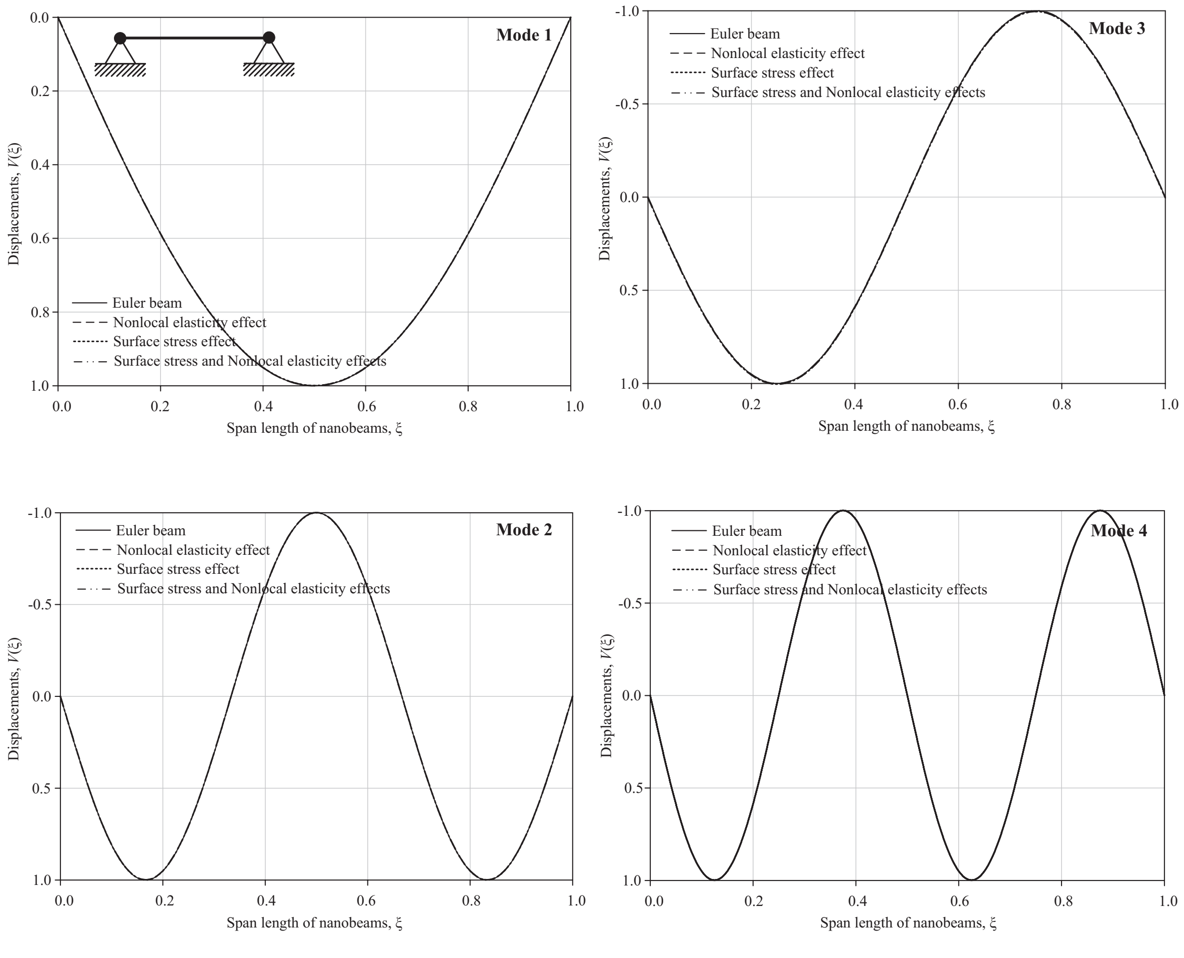

Fig. 3 Variations of mode shapes of pinned-pinned nanobeams

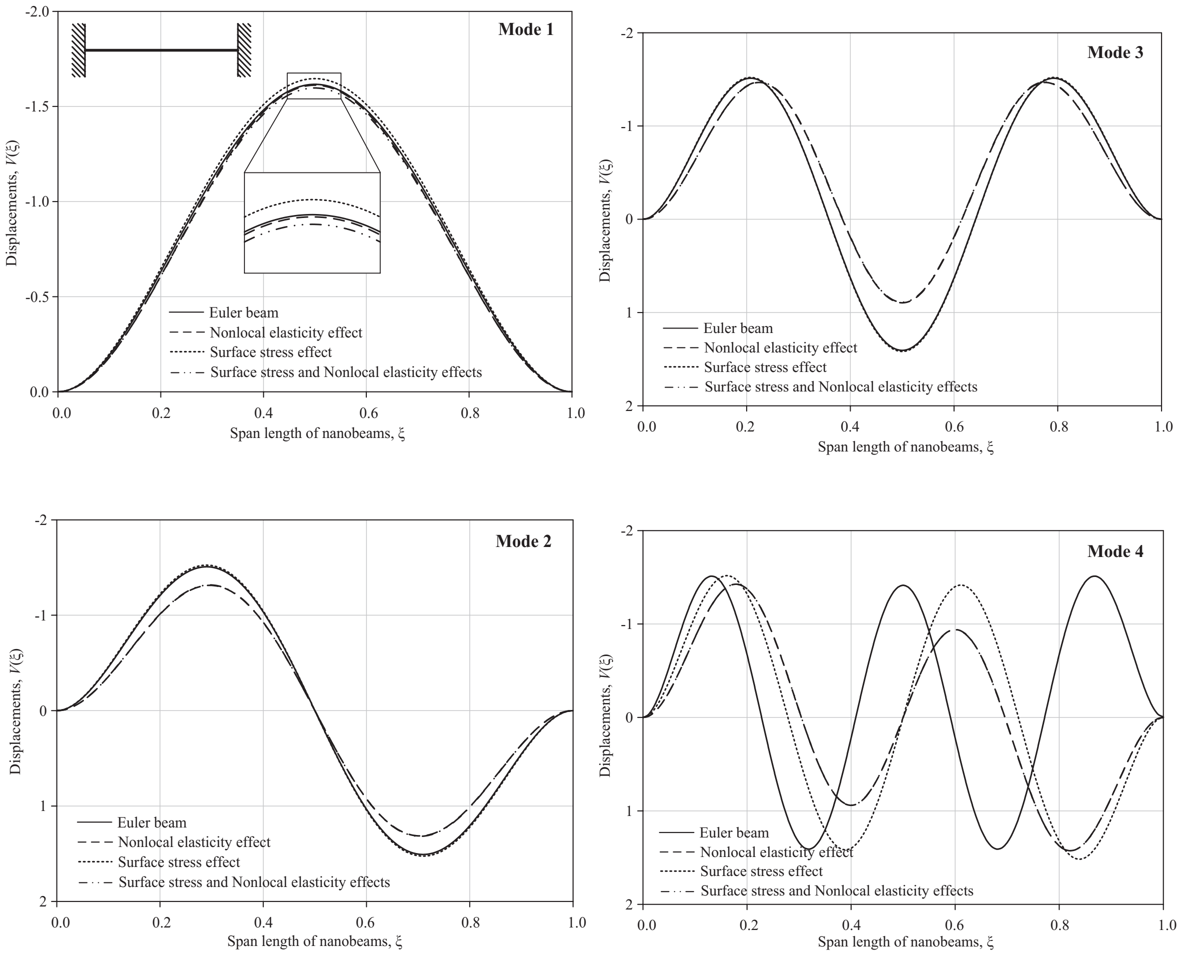

Fig. 4 Variations of mode shapes of clamped-clamped nanobeams.

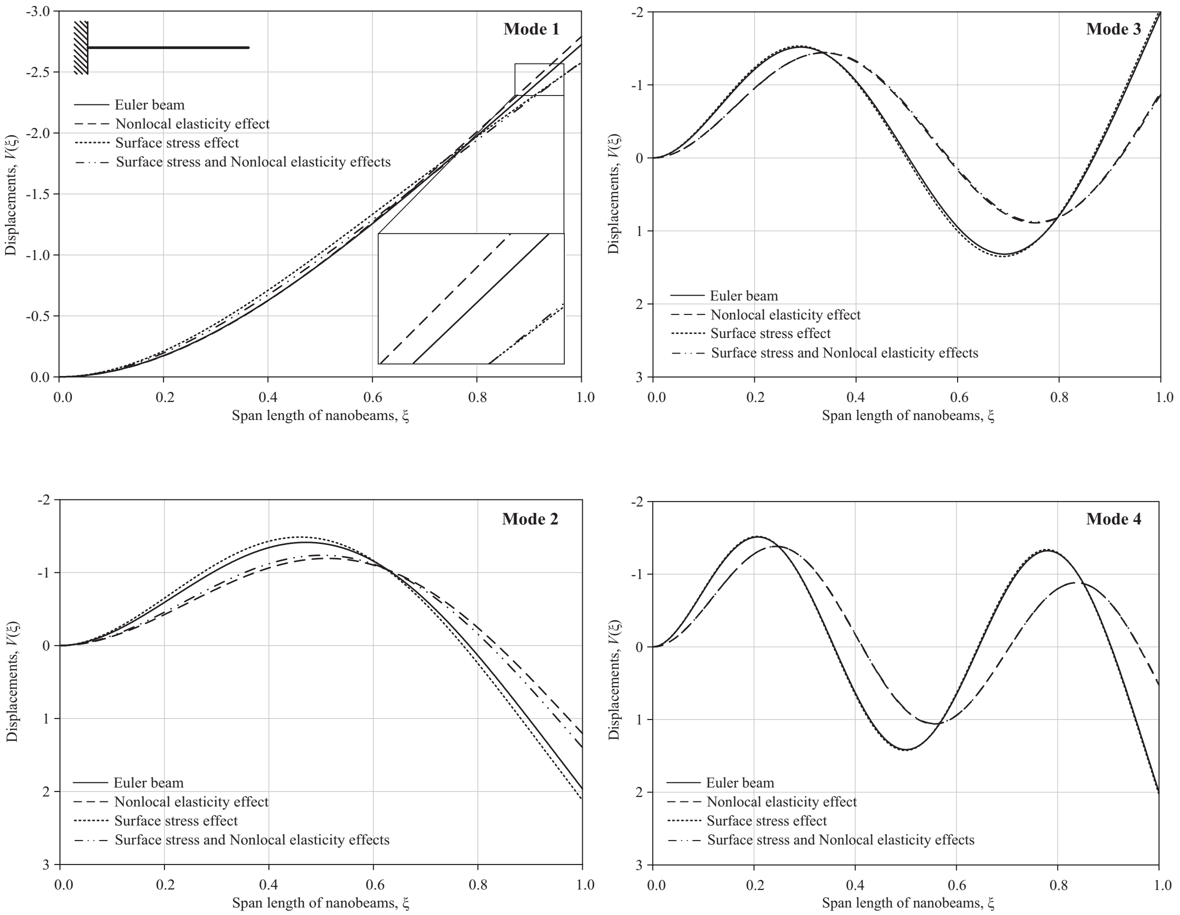

Fig. 5 Variations of mode shapes of clamped-free nanobeams.

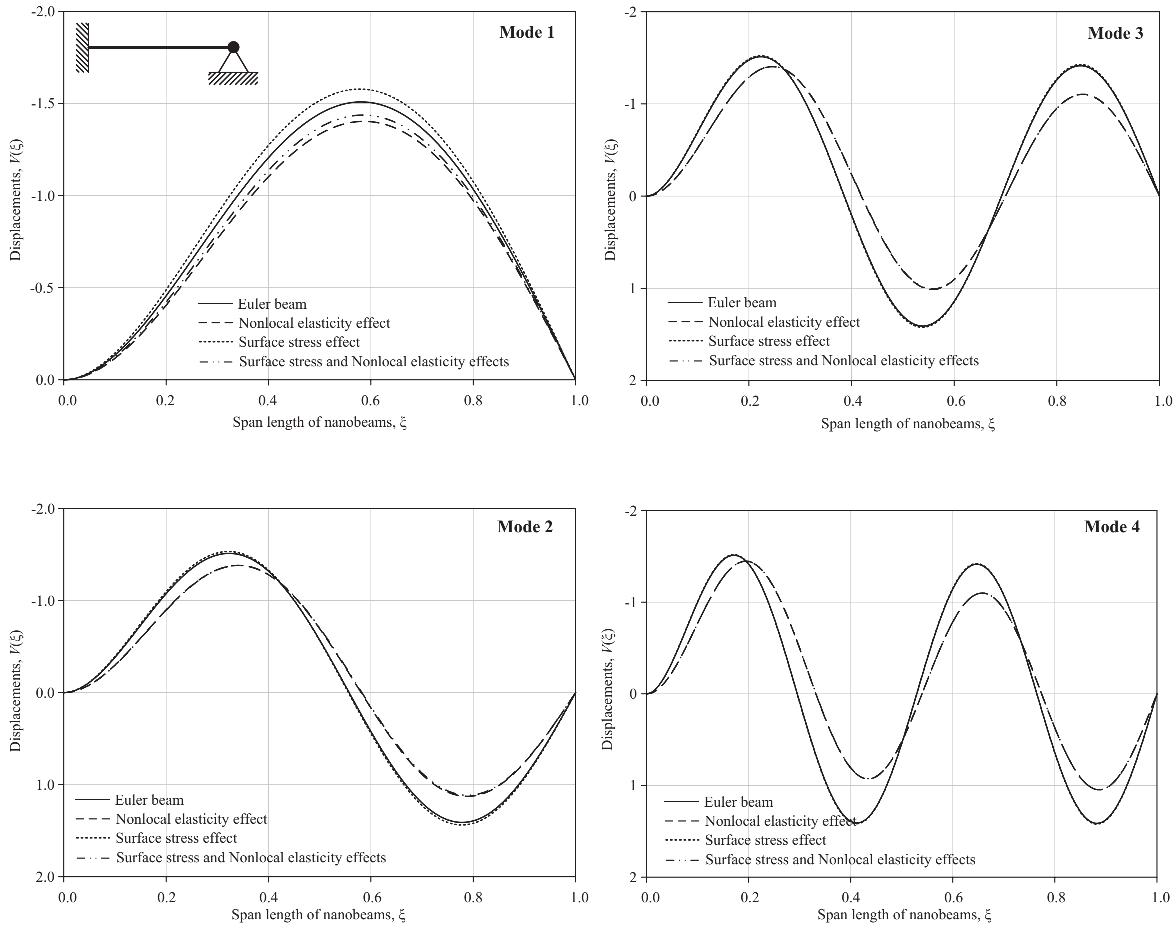

Fig. 6 Variations of mode shapes of clamped-pinned nanobeams.

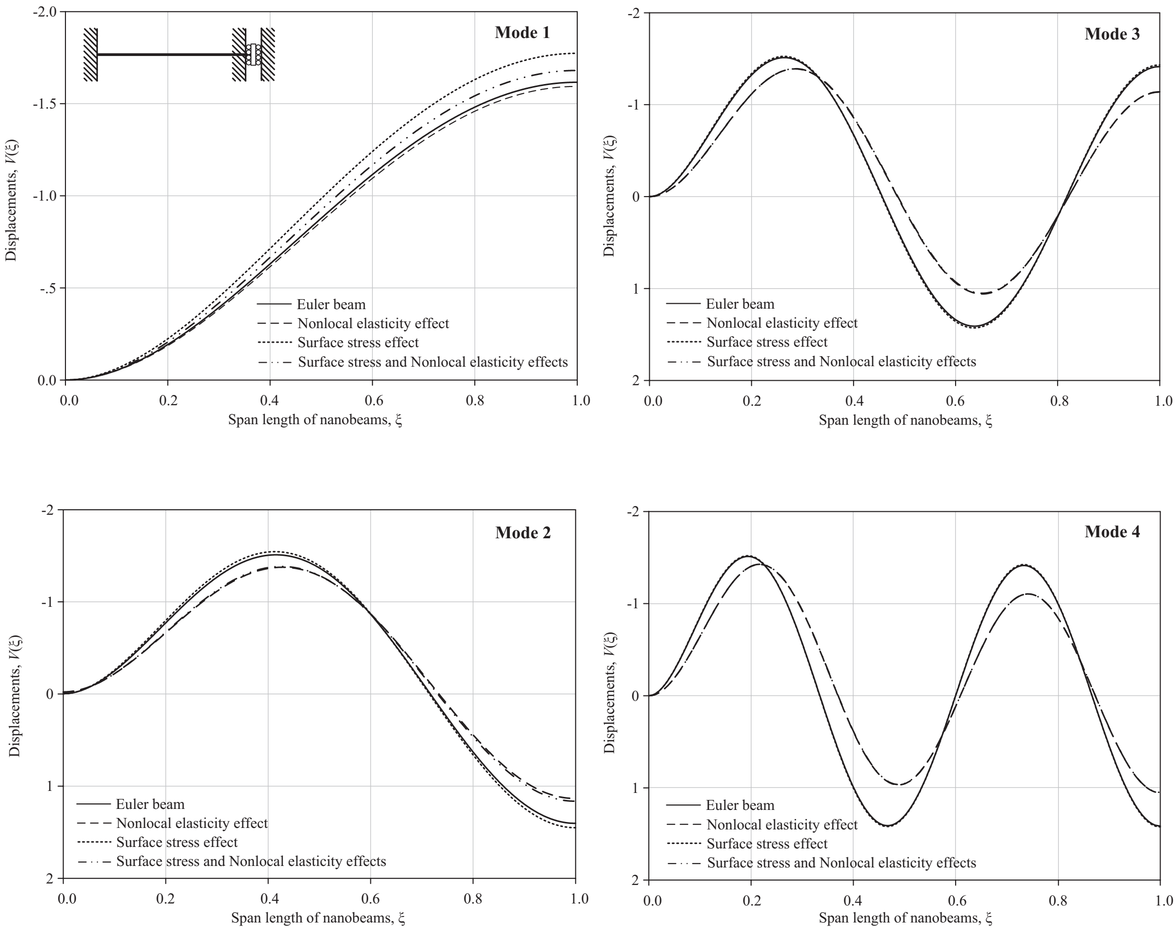

Fig. 7 Variations of mode shapes of clamped-sliding nanobeams.

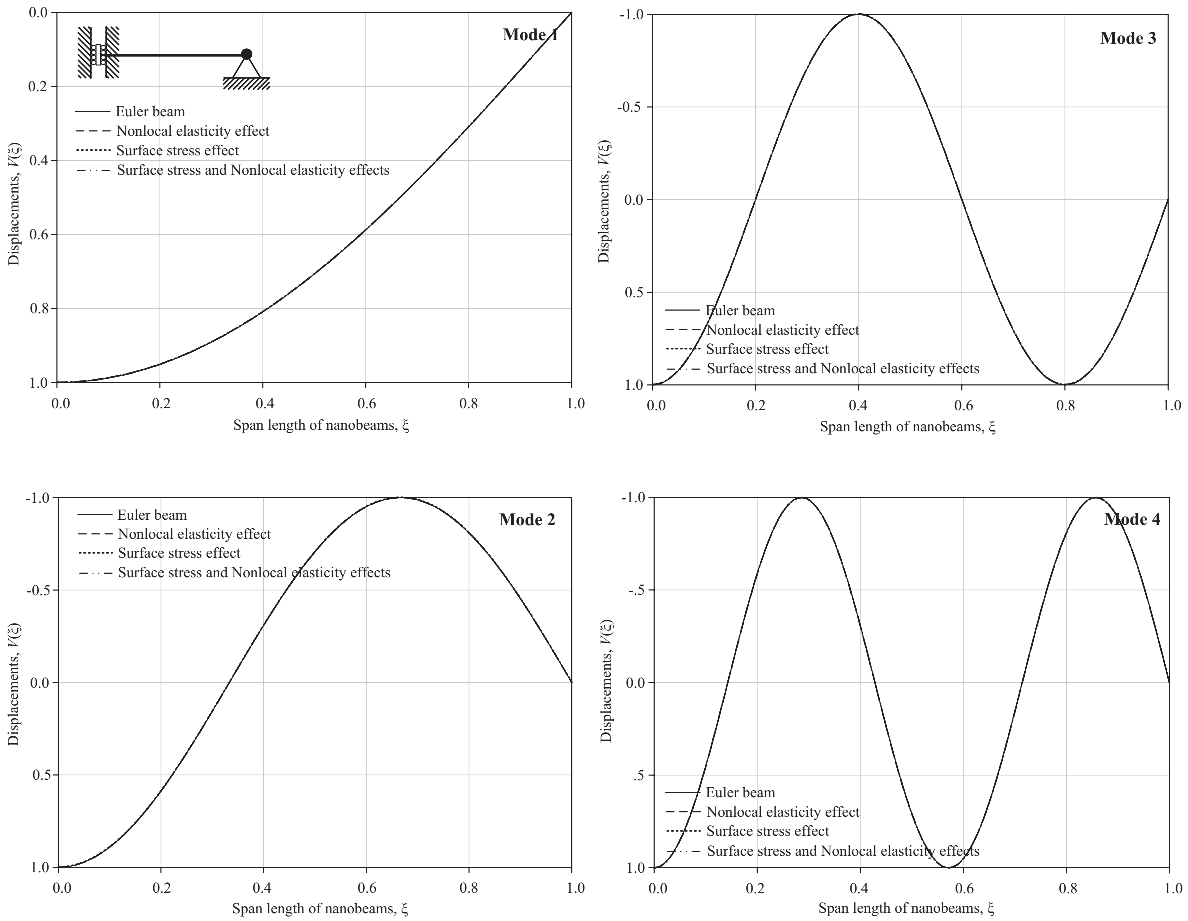

Fig. 8 Variations of mode shapes of sliding-pinned nanobeams.

Finally, the variation of dimensionless natural frequencies of nanobeams under various boundary conditions are illustrated as presented in Fig.9. The comparison between the nanobeams with surface stress effect and Euler beam, the nanobeams with non-local elasticity effect and Euler beam, and the nanobeams with combined effects of surface stress and non-local elasticity and Euler beam are also shown and found that the surface stress results the increment of natural frequencies but its influence is reduced and then has no effect when the mode number is higher than the fourth mode. The decrement of natural frequencies of nanobeams with accounting only surface stress obtained in this work has a similar trend in comparison with the work of Wang and Feng [3]. For the nanobeams including non-local elasticity, there can be seen that the natural frequencies of nanobeams for all boundary conditions are decreased significantly for all modes number. These obtained results give the similar behaviors as presented in the previous work [17]. It is also observed that the effect of the non-local elasticity increases for the higher modes of vibration. When the surface stress is combined with that of non-local elasticity, the obtained natural frequencies are in between the one of nanobeams with surface stress and the one with non-local elasticity. For the higher modes of vibration, the natural frequencies are converted to the nanobeams with non-local elasticity. The reasons are based on the fact that the influence of surface stress is negligible small for the modes number which is higher than the fourth mode.

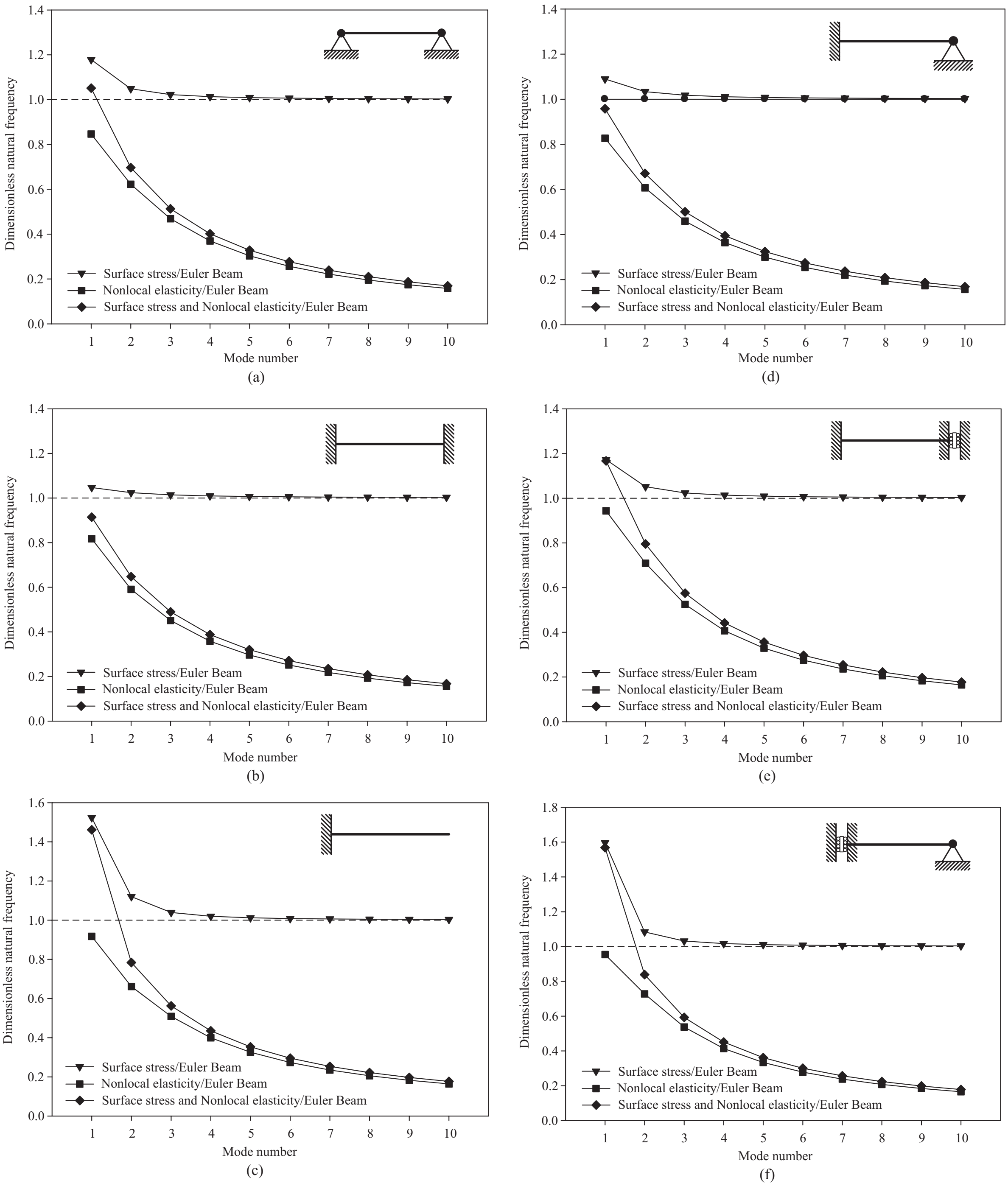

Fig. 9 Variation of dimensionless natural frequency with surface stress and non-local elasticity effects for different boundary condition: (a) pinned-pinned, (b) clamped-clamped, (c) clamped-free, (d) clamped-pinned, (e) clamped-sliding, and (f) sliding-pinned.

6. Conclusions

Analytical solutions of nanobeams including both surface stress and non-local elasticity for free vibration analysis are presented and verified numerically by using the finite element method. The analytical solutions show identical results with that of finite element method. The obtained results demonstrated that natural frequencies and corresponding modes shapes of nanobeams actually depend on the effects of surface stress and non-local elasticity. All of the results in this study are summarized as follows:

1. The surface stress increases the natural frequencies, especially, for lower modes number. The effect of surface stress is almost disappeared for the mode number with higher than the fourth mode. Moreover, non-local elasticity decreases the natural frequencies of nanobeams and its effect is increased for higher modes of vibration.

2. For the nanobeams including the combined effects of surface stress and non-local elasticity, the results indicate that the natural frequencies are in between the one of nanobeams with surface stress and the one with non-local elasticity. Since the surface effect is reduced for the higher modes number, the natural frequencies of nanobeams including combined effects are converted to nanobeams which only non-local elasticity is considered.

3. The clamped-clamped, clamped-free, clamped-pinned and clamped-sliding nanobeams exhibit the variation of mode shapes when the effects of surface stress and non-local elasticity are included. However, the surface stress and non-local elasticity have no effect in case of the pinned-pinned and sliding-pinned nanobeams.

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